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Cumulants, coherence, and contamination in multiparticle Bose-Einstein interferometry

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We examine the formalism of multiparticle correlations used in Bose-Einstein interferometry with pions produced in ultrarelativistic heavy ion collisions. We include incoherent and quantum optics coherent contributions as well as the effect of contamination from particles included in the correlation that are not pions. We give expressions for the correlation functions and normalized cumulants for orders 2–5 in the presence of these effects. We show that in the presence of coherence the normalized cumulants include an additional contribution besides that usually called the ''true'' multiparticle correlation. We also consider the $Q=0$ intercepts of the correlation functions and normalized cumulants in the presence of coherence and of contamination and show that values of the intercept of the normalized cumulant as a function of order can distinguish these two effects.

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I. INTRODUCTION

In Bose-Einstein correlation analysis of two or more pions produced in ultrarelativistic heavy ion collisions, an empirical parameter λ is frequently used to reduce the correlation function, in order to take into account the possibilities that (a) the pion emission from the source may not be completely incoherent, and (b) the correlated particles assumed to be pions may be contaminated with other particles (kaons, electrons, protons, etc.) which will dilute the measured correlations. However, these two effects have qualitatively different consequences for the magnitude and shape of the correlation and should be treated separately. In the present work we examine, for orders 2–5, the effects of both coherence and contamination on correlation functions and normalized cumulants used in the analysis of multiparticle Bose-Einstein correlations.

In the recent literature of multiparticle Bose-Einstein correlations of pions there has been considerable interest in isolating the ''true'' multiparticle correlations [1] that contribute to the overall correlation functions, e.g., correlations that are not representable as a product of lower-order correlations. For an incoherent source, this correlation arises from the simultaneous exchange of all particles in the correlated set. Such correlations must be present in an ideal quantum system of identical Bose-Einstein particles as a consequence of proper symmetrization of the multiparticle wave function, but demonstrating this has been an experimental challenge. Eggers *et al.* [2] have suggested using normalized cumulants to isolate the ''true'' multiparticle correlation.

In the present work we investigate the form of these normalized cumulants in the presence of coherent interference at the source of pions. As will be shown below, in the presence of coherence the normalized cumulants no longer isolate the ''true'' multiparticle correlation. We also consider the correspondence between the coherence terms calculated and the ''linked-pair'' approximation [3]. Finally, we consider the effects of coherence and contamination on the ''intercept'' values of correlations and normalized cumulants.

II. DEFINITIONS

A. Correlation functions

The multiparticle correlation functions R_n of order n used in pion interferometry are defined by the relation

$$R_n(\vec{P}_1,\ldots,\vec{P}_n)=\frac{\rho_n(\vec{P}_1,\ldots,\vec{P}_n)}{\rho_1(\vec{P}_1)\cdots\rho_1(\vec{P}_n)}, \tag{1}$$

where $\rho_n(\vec{P}_1,\ldots,\vec{P}_n)$ is the inclusive density for n particles expressed as a function of the three-momenta $\vec{P}_i$ of the correlated particles, and $\rho_1(\vec{P}_i)$ is the single-particle density of the ith particle.

In a previous paper [4], one of us has used the procedure of Biyajima *et al.* [5] based on diagrams from quantum optics, hereafter referred to as the Biyajima procedure, to derive general model-independent two-, three-, and four-particle correlation distributions for bosons which include the effects of coherence while neglecting Coulomb effects. The Biyajima procedure includes a coherent contribution to particle emission, but it implicitly assumes that there is *only one* source of coherent emission. In the present work we will focus on Bose-Einstein interferometry with pions, but we note that our conclusions also apply to interferometry with other bosons, e.g., kaons or photons.

The two-neutral-particle correlation function R_2, previously calculated using the Biyajima procedure [4,5], has the form

$$R_2(\vec{P}_1,\vec{P}_2)=1+\epsilon^2 b_{12}^2+2\,\phi\epsilon b_{12}. \tag{2}$$

Here the b_{ij} are two-particle Bose-Einstein exchange amplitudes, which in principle can be complex [6], with $b_{ij}=b_{ji}^*$. For the purposes of the present work, since $\mathrm{Im}(b_{ij})$ is usually small, we will take the b_{ij} amplitudes to be real, with $b_{ij}=b_{ji}$, and will consistently place the smallest index first. The results presented below can, however, be

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extended to the more general case of complex b_{ij} by taking all squares of b_{ij} to be absolute squares and then taking the real parts only of all terms involving b_{ij}.

We consider the total multiplicity $m_{\rm total}$ of final state particles used in the correlation to come from three sources: (1) incoherent production of pions ($m_{\rm inc}$), (2) coherent production of pions ($m_{\rm coh}$), and (3) particles treated as pions which are actually kaons, electrons, protons, etc. ($m_{\rm cont}$), so that $m_{\rm total}=m_{\rm inc}+m_{\rm coh}+m_{\rm cont}$. In the above expression, ϵ specifies the fraction of the net emission of the source that is incoherent, i.e., $\epsilon=m_{\rm inc}/m_{\rm total}$, and ϕ specifies the fraction of the net emission of the source that is coherent, i.e., $\phi=m_{\rm coh}/m_{\rm total}$. The contaminant fraction (not explicitly used here) is given by $\kappa=m_{\rm cont}/m_{\rm total}$. Thus, $\epsilon+\phi+\kappa=1$. We note that in general ϵ, ϕ, and κ are not constants and will be functions of the rapidity y and the transverse mass m_t of the reaction products. In the present work we will neglect any additional Bose-Einstein or Fermi-Dirac correlations between groups of identical contaminant particles because we expect their fraction to be small ($\kappa\approx 0.1$ or less).

The second term of Eq. (2), which depends on ϵ^2, results from the incoherence of the source of pions. The last term, which depends on $\phi\epsilon$, results from interference between the coherent and incoherent contributions to the process and is thus a cross term. There is no pure coherent term because complete coherence suppresses the Bose-Einstein correlation and gives a zero contribution to the correlation. When either coherent emission or contamination is significant, $\epsilon<1$ and, as will be discussed below, this reduces the peak of the correlation distribution and in particular its intercept value near $Q=0$ [where $Q^2=-(\mathscr{P}_i-\mathscr{P}_j)^2$ and $\mathscr{P}_i$ is a four-momentum].

Before proceeding to higher-order correlations, let us consider two examples of the use of Eq. (2). First, let us consider a case where there is a contribution from coherence but none from contamination, i.e., $\epsilon=\sqrt{\lambda}$, $\phi=(1-\epsilon)$, $\kappa=0$, and $b_{12}=\exp[-(q_{12}r)^2]$, where q_{12} is the magnitude of the three-momentum difference between the correlated particles, r is the source radius, and λ is an "intercept" parameter permitting adjustment of the strength of the correlation. Then Eq. (2) becomes

$$R_2(q_{12};\kappa=0)=1+\lambda\exp[-2\,(q_{12}r)^2] + 2\,\lambda^{1/2}(1-\lambda^{1/2})\exp[-(q_{12}r)^2]. \quad (3)$$

The first two terms of Eq. (3) are the usual empirical Gaussian representation of a two-particle Bose-Einstein correlation, while the last term is an additional one reflecting the effects of coherence. Note that, as previously pointed out by Weiner [7], the two Gaussians in Eq. (3) have widths which differ by a factor of $\sqrt{2}$, so that the onset of coherence tends to broaden a measured momentum-space correlation function.

Now let us consider a case where there is a contribution from contamination but none from coherence, i.e., $\epsilon=\sqrt{\lambda}$, $\phi=0$, $\kappa=(1-\epsilon)$, and as before $b_{12}=\exp[-(q_{12}r)^2]$. With these assumptions, Eq. (2) becomes

$$R_2(q_{12};\phi=0)=1+\lambda\exp[-2\,(q_{12}r)^2]. \quad (4)$$

The first two terms of Eq. (4) are the same as those of Eq. (3), but the last term is missing. This is because coherence produces interference effects, while contamination produces none.

The three neutral particle correlation function R_3 derived from the Biyajima procedure [5] is

$$R_3(\vec{P}_1,\vec{P}_2,\vec{P}_3)=1+\epsilon^2(b_{12}^2+b_{23}^2+b_{13}^2)+2\epsilon^3(b_{12}b_{23}b_{13}) + 2\,\phi\epsilon(b_{12}+b_{23}+b_{13})+2\,\phi\epsilon^2(b_{12}b_{23} + b_{23}b_{13}+b_{13}b_{12}). \quad (5)$$

The second and third terms of Eq. (5) depend on powers of ϵ and result from source incoherence. The last two terms depend on powers of ϕ and ϵ and result from coherent-incoherent interference.

For four neutral particles, the correlation function R_4 as rederived here from the Biyajima procedure is

$$\begin{aligned}R_4(\vec{P}_1,\vec{P}_2,\vec{P}_3,\vec{P}_4)={}&1+\epsilon^2(b_{12}^2+b_{13}^2+b_{14}^2+b_{23}^2+b_{24}^2+b_{34}^2)+2\epsilon^3(b_{12}b_{23}b_{13}+b_{12}b_{24}b_{14}+b_{13}b_{34}b_{14}+b_{23}b_{34}b_{24})\\&+2\epsilon^4(b_{12}b_{23}b_{34}b_{14}+b_{12}b_{24}b_{34}b_{13}+b_{13}b_{23}b_{24}b_{14})+\epsilon^4(b_{12}^2b_{34}^2+b_{13}^2b_{24}^2+b_{14}^2b_{23}^2)+2\,\phi\epsilon(b_{12}+b_{13}\\&+b_{14}+b_{23}+b_{24}+b_{34})+2\,\phi\epsilon^2(b_{12}b_{23}+b_{12}b_{24}+b_{13}b_{34}+b_{23}b_{34}+b_{12}b_{13}+b_{12}b_{14}+b_{13}b_{14}+b_{23}b_{24}\\&+b_{13}b_{23}+b_{14}b_{24}+b_{14}b_{34}+b_{24}b_{34})+4\,\phi^2\epsilon^2(b_{12}b_{34}+b_{13}b_{24}+b_{14}b_{23})+2\,\phi\epsilon^3(b_{12}b_{23}b_{34}+b_{12}b_{24}b_{34}\\&+b_{13}b_{23}b_{24}+b_{12}b_{23}b_{14}+b_{12}b_{24}b_{13}+b_{13}b_{23}b_{14}+b_{12}b_{34}b_{14}+b_{12}b_{34}b_{13}+b_{13}b_{24}b_{14}+b_{23}b_{34}b_{14}\\&+b_{24}b_{34}b_{13}+b_{23}b_{24}b_{14})+2\,\phi\epsilon^3(b_{12}b_{34}^2+b_{13}b_{24}^2+b_{14}b_{23}^2+b_{12}^2b_{34}+b_{13}^2b_{24}+b_{14}^2b_{23}). \quad (6)\end{aligned}$$

Again, the second through fifth terms of Eq. (6) depend on powers of ϵ and result from source incoherence. The last five terms depend on powers of ϵ and ϕ and result from coherent-incoherent interference. For a completely incoherent system ($\epsilon=1$) with no contamination, the ϕ terms in the above relations will vanish and the expressions will be considerably simplified. For a completely coherent systems ($\phi=1$), all terms will vanish except the leading 1. We note that the seventh term of Eq. (5), which depends on $2\phi\epsilon^2$, differs from the similar term in Eq. (22) of Ref. [4] and corrects a subtle error in that paper.

We have derived R_5, the general correlation function for five neutral particles with the Biyajima procedure, using MATHEMATICA [8] and present it here for the first time:

$$
\begin{aligned}
2R_5(\vec{P}_1,\vec{P}_2,\vec{P}_3,\vec{P}_4,\vec{P}_5)={}&1+2\,\epsilon^5(b_{14}b_{15}b_{23}b_{25}b_{34}+b_{13}b_{15}b_{24}b_{25}b_{34}+b_{14}b_{15}b_{23}b_{24}b_{35}+b_{13}b_{14}b_{24}b_{25}b_{35}+b_{12}b_{15}b_{24}b_{34}b_{35}\\
&+b_{12}b_{14}b_{25}b_{34}b_{35}+b_{13}b_{15}b_{23}b_{24}b_{45}+b_{13}b_{14}b_{23}b_{25}b_{45}+b_{12}b_{15}b_{23}b_{34}b_{45}+b_{12}b_{13}b_{25}b_{34}b_{45}\\
&+b_{12}b_{14}b_{23}b_{35}b_{45}+b_{12}b_{13}b_{24}b_{35}b_{45})+2\,\epsilon^5(b_{15}^2b_{23}b_{24}b_{34}+b_{13}b_{14}b_{25}^2b_{34}+b_{12}b_{15}b_{25}b_{34}^2\\
&+b_{13}b_{15}b_{24}^2b_{35}+b_{14}^2b_{23}b_{25}b_{35}+b_{12}b_{14}b_{24}b_{35}^2+b_{14}b_{15}b_{23}^2b_{45}+b_{13}^2b_{24}b_{25}b_{45}+b_{12}^2b_{34}b_{35}b_{45}\\
&+b_{12}b_{13}b_{23}b_{45}^2)+2\,\epsilon^4(b_{13}b_{14}b_{23}b_{24}+b_{13}b_{15}b_{23}b_{25}+b_{14}b_{15}b_{24}b_{25}+b_{12}b_{14}b_{23}b_{34}+b_{12}b_{13}b_{24}b_{34}\\
&+b_{12}b_{15}b_{23}b_{35}+b_{12}b_{13}b_{25}b_{35}+b_{14}b_{15}b_{34}b_{35}+b_{24}b_{25}b_{34}b_{35}+b_{12}b_{15}b_{24}b_{45}+b_{12}b_{14}b_{25}b_{45}\\
&+b_{13}b_{15}b_{34}b_{45}+b_{23}b_{25}b_{34}b_{45}+b_{13}b_{14}b_{35}b_{45}+b_{23}b_{24}b_{35}b_{45})+\epsilon^4(b_{14}^2b_{23}^2+b_{15}^2b_{23}^2+b_{13}^2b_{24}^2\\
&+b_{15}^2b_{24}^2+b_{13}^2b_{25}^2+b_{14}^2b_{25}^2+b_{12}^2b_{34}^2+b_{15}^2b_{34}^2+b_{25}^2b_{34}^2+b_{12}^2b_{35}^2+b_{14}^2b_{35}^2+b_{24}^2b_{35}^2+b_{12}^2b_{45}^2+b_{13}^2b_{45}^2\\
&+b_{23}^2b_{45}^2)+2\,\epsilon^3(b_{12}b_{13}b_{23}+b_{12}b_{14}b_{24}+b_{12}b_{15}b_{25}+b_{13}b_{14}b_{34}+b_{23}b_{24}b_{34}+b_{13}b_{15}b_{35}+b_{23}b_{25}b_{35}\\
&+b_{14}b_{15}b_{45}+b_{24}b_{25}b_{45}+b_{34}b_{35}b_{45})+\epsilon^2(b_{12}^2+b_{13}^2+b_{14}^2+b_{15}^2+b_{23}^2+b_{24}^2+b_{25}^2+b_{34}^2+b_{35}^2+b_{45}^2)\\
&+2\,\phi\epsilon(b_{12}+b_{13}+b_{14}+b_{15}+b_{23}+b_{24}+b_{25}+b_{34}+b_{35}+b_{45})+4\,\phi^2\epsilon^2(b_{14}b_{23}+b_{15}b_{23}+b_{13}b_{24}\\
&+b_{15}b_{24}+b_{13}b_{25}+b_{14}b_{25}+b_{12}b_{34}+b_{15}b_{34}+b_{25}b_{34}+b_{12}b_{35}+b_{14}b_{35}+b_{24}b_{35}+b_{12}b_{45}+b_{13}b_{45}\\
&+b_{23}b_{45})+2\,\phi\epsilon^2(b_{12}b_{13}+b_{12}b_{14}+b_{13}b_{14}+b_{12}b_{15}+b_{13}b_{15}+b_{14}b_{15}+b_{12}b_{23}+b_{13}b_{23}+b_{12}b_{24}\\
&+b_{14}b_{24}+b_{23}b_{24}+b_{12}b_{25}+b_{15}b_{25}+b_{23}b_{25}+b_{24}b_{25}+b_{13}b_{34}+b_{14}b_{34}+b_{23}b_{34}+b_{24}b_{34}+b_{13}b_{35}\\
&+b_{15}b_{35}+b_{23}b_{35}+b_{25}b_{35}+b_{34}b_{35}+b_{14}b_{45}+b_{15}b_{45}+b_{24}b_{45}+b_{25}b_{45}+b_{34}b_{45}+b_{35}b_{45})\\
&+4\,\phi^2\epsilon^3(b_{14}b_{15}b_{23}+b_{13}b_{15}b_{24}+b_{15}b_{23}b_{24}+b_{13}b_{14}b_{25}+b_{14}b_{23}b_{25}+b_{13}b_{24}b_{25}+b_{12}b_{15}b_{34}\\
&+b_{15}b_{23}b_{34}+b_{15}b_{24}b_{34}+b_{12}b_{25}b_{34}+b_{13}b_{25}b_{34}+b_{14}b_{25}b_{34}+b_{15}b_{25}b_{34}+b_{12}b_{14}b_{35}+b_{14}b_{23}b_{35}\\
&+b_{12}b_{24}b_{35}+b_{13}b_{24}b_{35}+b_{14}b_{24}b_{35}+b_{15}b_{24}b_{35}+b_{14}b_{25}b_{35}+b_{12}b_{34}b_{35}+b_{12}b_{13}b_{45}+b_{12}b_{23}b_{45}\\
&+b_{13}b_{23}b_{45}+b_{14}b_{23}b_{45}+b_{15}b_{23}b_{45}+b_{13}b_{24}b_{45}+b_{13}b_{25}b_{45}+b_{12}b_{34}b_{45}+b_{12}b_{35}b_{45})\\
&+2\,\phi\epsilon^3(b_{12}b_{14}b_{23}+b_{13}b_{14}b_{23}+b_{12}b_{15}b_{23}+b_{13}b_{15}b_{23}+b_{12}b_{13}b_{24}+b_{13}b_{14}b_{24}+b_{12}b_{15}b_{24}\\
&+b_{14}b_{15}b_{24}+b_{13}b_{23}b_{24}+b_{14}b_{23}b_{24}+b_{12}b_{13}b_{25}+b_{12}b_{14}b_{25}+b_{13}b_{15}b_{25}+b_{14}b_{15}b_{25}+b_{13}b_{23}b_{25}\\
&+b_{15}b_{23}b_{25}+b_{14}b_{24}b_{25}+b_{15}b_{24}b_{25}+b_{12}b_{13}b_{34}+b_{12}b_{14}b_{34}+b_{13}b_{15}b_{34}+b_{14}b_{15}b_{34}+b_{12}b_{23}b_{34}\\
&+b_{14}b_{23}b_{34}+b_{12}b_{24}b_{34}+b_{13}b_{24}b_{34}+b_{23}b_{25}b_{34}+b_{24}b_{25}b_{34}+b_{12}b_{13}b_{35}+b_{13}b_{14}b_{35}+b_{12}b_{15}b_{35}\\
&+b_{14}b_{15}b_{35}+b_{12}b_{23}b_{35}+b_{15}b_{23}b_{35}+b_{23}b_{24}b_{35}+b_{12}b_{25}b_{35}+b_{13}b_{25}b_{35}+b_{24}b_{25}b_{35}+b_{14}b_{34}b_{35}\\
&+b_{15}b_{34}b_{35}+b_{24}b_{34}b_{35}+b_{25}b_{34}b_{35}+b_{12}b_{14}b_{45}+b_{13}b_{14}b_{45}+b_{12}b_{15}b_{45}+b_{13}b_{15}b_{45}+b_{15}b_{24}b_{45}\\
&+b_{23}b_{24}b_{45}+b_{12}b_{25}b_{45}+b_{14}b_{25}b_{45}+b_{23}b_{25}b_{45}+b_{13}b_{34}b_{45}+b_{15}b_{34}b_{45}+b_{23}b_{34}b_{45}+b_{25}b_{34}b_{45}\\
&+b_{13}b_{35}b_{45}+b_{14}b_{35}b_{45}+b_{23}b_{35}b_{45}+b_{12}b_{24}b_{45}+b_{24}b_{35}b_{45})+2\,\phi\epsilon^3(b_{14}^2b_{23}+b_{15}^2b_{23}+b_{14}b_{23}^2\\
&+b_{15}b_{23}^2+b_{13}^2b_{24}+b_{15}^2b_{24}+b_{13}b_{24}^2+b_{15}b_{24}^2+b_{13}^2b_{25}+b_{14}^2b_{25}+b_{13}b_{25}^2+b_{14}b_{25}^2+b_{12}^2b_{34}+b_{15}^2b_{34}\\
&+b_{12}^2b_{35}+b_{14}^2b_{35}+b_{25}^2b_{34}+b_{12}b_{34}^2+b_{15}b_{34}^2+b_{25}b_{34}^2+b_{24}^2b_{35}+b_{12}b_{35}^2+b_{14}b_{35}^2+b_{24}b_{35}^2+b_{12}^2b_{45}\\
&+b_{13}^2b_{45}+b_{23}^2b_{45}+b_{12}b_{45}^2+b_{13}b_{45}^2+b_{23}b_{45}^2)+2\,\phi\epsilon^4(b_{14}b_{15}b_{23}^2+b_{15}^2b_{23}b_{24}+b_{13}b_{15}b_{24}^2+b_{14}^2b_{23}b_{25}\\
&+b_{13}^2b_{24}b_{25}+b_{13}b_{14}b_{25}^2+b_{15}^2b_{23}b_{34}+b_{15}^2b_{24}b_{34}+b_{13}b_{25}^2b_{34}+b_{14}b_{25}^2b_{34}+b_{12}b_{15}b_{34}^2+b_{12}b_{25}b_{34}^2\\
&+b_{15}b_{25}b_{34}^2+b_{14}^2b_{23}b_{35}+b_{13}b_{24}^2b_{35}+b_{15}b_{24}^2b_{35}+b_{14}^2b_{25}b_{35}+b_{12}^2b_{34}b_{35}+b_{12}b_{14}b_{35}^2+b_{12}b_{24}b_{35}^2\\
&+b_{14}b_{24}b_{35}^2+b_{14}b_{23}^2b_{45}+b_{15}b_{23}^2b_{45}+b_{13}^2b_{24}b_{45}+b_{13}^2b_{25}b_{45}+b_{12}^2b_{34}b_{45}+b_{12}^2b_{35}b_{45}+b_{12}b_{13}b_{45}^2\\
&+b_{12}b_{23}b_{45}^2+b_{13}b_{23}b_{45}^2)+2\,\phi\epsilon^4(b_{13}b_{15}b_{23}b_{24}+b_{14}b_{15}b_{23}b_{24}+b_{13}b_{14}b_{23}b_{25}+b_{14}b_{15}b_{23}b_{25}\\
&+b_{13}b_{14}b_{24}b_{25}+b_{13}b_{15}b_{24}b_{25}+b_{12}b_{15}b_{23}b_{34}+b_{14}b_{15}b_{23}b_{34}+b_{12}b_{15}b_{24}b_{34}+b_{13}b_{15}b_{24}b_{34}\\
&+b_{12}b_{13}b_{25}b_{34}+b_{12}b_{14}b_{25}b_{34}+b_{13}b_{15}b_{25}b_{34}+b_{14}b_{15}b_{25}b_{34}+b_{14}b_{23}b_{25}b_{34}+b_{15}b_{23}b_{25}b_{34}
\end{aligned}
$$

$$+b_{13}b_{24}b_{25}b_{34}+b_{15}b_{24}b_{25}b_{34}+b_{12}b_{14}b_{23}b_{35}+b_{14}b_{15}b_{23}b_{35}+b_{12}b_{13}b_{24}b_{35}+b_{13}b_{14}b_{24}b_{35}$$
$$+b_{12}b_{15}b_{24}b_{35}+b_{14}b_{15}b_{24}b_{35}+b_{14}b_{23}b_{24}b_{35}+b_{15}b_{23}b_{24}b_{35}+b_{12}b_{14}b_{25}b_{35}+b_{13}b_{14}b_{25}b_{35}$$
$$+b_{13}b_{24}b_{25}b_{35}+b_{14}b_{24}b_{25}b_{35}+b_{12}b_{14}b_{34}b_{35}+b_{12}b_{15}b_{34}b_{35}+b_{12}b_{24}b_{34}b_{35}+b_{15}b_{24}b_{34}b_{35}$$
$$+b_{12}b_{25}b_{34}b_{35}+b_{14}b_{25}b_{34}b_{35}+b_{12}b_{14}b_{23}b_{45}+b_{13}b_{14}b_{23}b_{45}+b_{12}b_{15}b_{23}b_{45}+b_{13}b_{15}b_{23}b_{45}$$
$$+b_{12}b_{13}b_{24}b_{45}+b_{13}b_{15}b_{24}b_{45}+b_{13}b_{23}b_{24}b_{45}+b_{15}b_{23}b_{24}b_{45}+b_{12}b_{13}b_{25}b_{45}+b_{13}b_{14}b_{25}b_{45}$$
$$+b_{13}b_{23}b_{25}b_{45}+b_{14}b_{23}b_{25}b_{45}+b_{12}b_{13}b_{34}b_{45}+b_{12}b_{15}b_{34}b_{45}+b_{12}b_{23}b_{34}b_{45}+b_{15}b_{23}b_{34}b_{45}$$
$$+b_{12}b_{25}b_{34}b_{45}+b_{13}b_{25}b_{34}b_{45}+b_{12}b_{13}b_{35}b_{45}+b_{12}b_{14}b_{35}b_{45}+b_{12}b_{23}b_{35}b_{45}+b_{14}b_{23}b_{35}b_{45}$$
$$+b_{12}b_{24}b_{35}b_{45}+b_{13}b_{24}b_{35}b_{45})+4\,\phi\epsilon^4(b_{15}b_{23}b_{24}b_{34}+b_{13}b_{14}b_{25}b_{34}+b_{12}b_{15}b_{25}b_{34}+b_{12}b_{14}b_{24}b_{35}$$
$$+b_{13}b_{15}b_{24}b_{35}+b_{14}b_{23}b_{25}b_{35}+b_{12}b_{13}b_{23}b_{45}+b_{14}b_{15}b_{23}b_{45}+b_{13}b_{24}b_{25}b_{45}+b_{12}b_{34}b_{35}b_{45}). \tag{7}$$

Again, the second through seventh terms of Eq. (7) depend on powers of ϵ and result from source incoherence. The last nine terms depend on powers of ϕ and ϵ and result from coherent-incoherent interference.

B. Cumulants

The cumulant [9] of a given order n is a combination of inclusive densities ρ_i with $i=1,\ldots,n$ constructed in such a way as to become zero whenever any one of their arguments becomes statistically independent of the others. The first two such cumulants have the forms

$$C_2(\vec{P}_1,\vec{P}_2)=\rho_2(\vec{P}_1,\vec{P}_2)-\rho_1(\vec{P}_1)\rho_1(\vec{P}_2), \tag{8}$$

$$C_3(\vec{P}_1,\vec{P}_2,\vec{P}_3)=\rho_3(\vec{P}_1,\vec{P}_2,\vec{P}_3)-\rho_1(\vec{P}_1)\rho_2(\vec{P}_2,\vec{P}_3)-\rho_1(\vec{P}_2)\rho_2(\vec{P}_3,\vec{P}_1)-\rho_1(\vec{P}_3)\rho_2(\vec{P}_1,\vec{P}_2)+2\,\rho_1(\vec{P}_1)\rho_1(\vec{P}_2)\rho_1(\vec{P}_3). \tag{9}$$

Omitting the momentum arguments, the next two higher cumulants are

$$C_4=\rho_4-\sum_{[4]}\rho_1\rho_3-\sum_{[3]}\rho_2\rho_2+2\sum_{[6]}\rho_1\rho_1\rho_2-6\,\rho_1\rho_1\rho_1\rho_1, \tag{10}$$

$$C_5=\rho_5-\sum_{[5]}\rho_1\rho_4-\sum_{[10]}\rho_2\rho_3+2\sum_{[10]}\rho_1\rho_1\rho_3+2\sum_{[15]}\rho_1\rho_2\rho_2-6\sum_{[10]}\rho_1\rho_1\rho_1\rho_2+24\,\rho_1\rho_1\rho_1\rho_1\rho_1. \tag{11}$$

Here the bracketed numbers under the sums indicate the number of permutations of the arguments $\vec{P}_i$ which have to be included in the sum.

C. Normalized cumulants

The normalized cumulant k_n of order n is generated by dividing the corresponding cumulant C_n defined above by the product of n single-particle density functions. It is therefore defined as

$$k_n=\frac{C_n(\vec{P}_1,\ldots,\vec{P}_n)}{\rho_1(\vec{P}_1),\ldots,\rho_1(\vec{P}_n)}. \tag{12}$$

These normalized cumulants can thus be written in terms of the correlation functions R_i defined above:

$$k_2(\vec{P}_1,\vec{P}_2)=R_2(\vec{P}_1,\vec{P}_2)-1, \tag{13}$$

$$\begin{aligned}k_3(\vec{P}_1,\vec{P}_2,\vec{P}_3)&=R_3(\vec{P}_1,\vec{P}_2,\vec{P}_3)+2-R_2(\vec{P}_2,\vec{P}_3)-R_2(\vec{P}_1,\vec{P}_3)-R_2(\vec{P}_1,\vec{P}_2)\\&=R_3(\vec{P}_1,\vec{P}_2,\vec{P}_3)-1-k_2(\vec{P}_2,\vec{P}_3)-k_2(\vec{P}_1,\vec{P}_3)-k_2(\vec{P}_1,\vec{P}_2),\end{aligned} \tag{14}$$

$$\begin{aligned}
k_4(\vec{P}_1,\vec{P}_2,\vec{P}_3,\vec{P}_4)=&R_4(\vec{P}_1,\vec{P}_2,\vec{P}_3,\vec{P}_4)-6-[R_3(\vec{P}_1,\vec{P}_2,\vec{P}_3)+R_3(\vec{P}_1,\vec{P}_2,\vec{P}_4)+R_3(\vec{P}_1,\vec{P}_3,\vec{P}_4)+R_3(\vec{P}_2,\vec{P}_3,\vec{P}_4)]\\
&-[R_2(\vec{P}_1,\vec{P}_2)R_2(\vec{P}_3,\vec{P}_4)+R_2(\vec{P}_1,\vec{P}_3)R_2(\vec{P}_2,\vec{P}_4)+R_2(\vec{P}_1,\vec{P}_4)R_2(\vec{P}_2,\vec{P}_3)]+2[R_2(\vec{P}_1,\vec{P}_2)\\
&+R_2(\vec{P}_1,\vec{P}_3)+R_2(\vec{P}_1,\vec{P}_4)+R_2(\vec{P}_2,\vec{P}_3)+R_2(\vec{P}_2,\vec{P}_4)+R_2(\vec{P}_3,\vec{P}_4)]\\
=&R_4(\vec{P}_1,\vec{P}_2,\vec{P}_3,\vec{P}_4)-1-[k_3(\vec{P}_1,\vec{P}_2,\vec{P}_3)+k_3(\vec{P}_1,\vec{P}_2,\vec{P}_4)+k_3(\vec{P}_1,\vec{P}_3,\vec{P}_4)+k_3(\vec{P}_2,\vec{P}_3,\vec{P}_4)]\\
&-[k_2(\vec{P}_1,\vec{P}_2)k_2(\vec{P}_3,\vec{P}_4)+k_2(\vec{P}_1,\vec{P}_3)k_2(\vec{P}_2,\vec{P}_4)+k_2(\vec{P}_1,\vec{P}_4)k_2(\vec{P}_2,\vec{P}_3)]-[k_2(\vec{P}_1,\vec{P}_2)+k_2(\vec{P}_1,\vec{P}_3)\\
&+k_2(\vec{P}_1,\vec{P}_4)+k_2(\vec{P}_2,\vec{P}_3)+k_2(\vec{P}_2,\vec{P}_4)+k_2(\vec{P}_3,\vec{P}_4)],
\end{aligned} \tag{15}$$

$$\begin{aligned}
k_5(\vec{P}_1,\vec{P}_2,\vec{P}_3,\vec{P}_4,\vec{P}_5)=&R_5(\vec{P}_1,\vec{P}_2,\vec{P}_3,\vec{P}_4,\vec{P}_5)+24-[R_4(\vec{P}_1,\vec{P}_2,\vec{P}_3,\vec{P}_4)+R_4(\vec{P}_1,\vec{P}_2,\vec{P}_3,\vec{P}_5)+R_4(\vec{P}_1,\vec{P}_2,\vec{P}_4,\vec{P}_5)\\
&+R_4(\vec{P}_1,\vec{P}_3,\vec{P}_4,\vec{P}_5)+R_4(\vec{P}_2,\vec{P}_3,\vec{P}_4,\vec{P}_5)]+2[R_3(\vec{P}_1,\vec{P}_2,\vec{P}_3)+R_3(\vec{P}_1,\vec{P}_2,\vec{P}_4)+R_3(\vec{P}_1,\vec{P}_2,\vec{P}_5)\\
&+R_3(\vec{P}_1,\vec{P}_3,\vec{P}_4)+R_3(\vec{P}_1,\vec{P}_3,\vec{P}_5)+R_3(\vec{P}_1,\vec{P}_4,\vec{P}_5)+R_3(\vec{P}_2,\vec{P}_3,\vec{P}_4)+R_3(\vec{P}_2,\vec{P}_3,\vec{P}_5)\\
&+R_3(\vec{P}_2,\vec{P}_4,\vec{P}_5)+R_3(\vec{P}_3,\vec{P}_4,\vec{P}_5)]-[R_3(\vec{P}_1,\vec{P}_2,\vec{P}_3)R_2(\vec{P}_4,\vec{P}_5)+R_3(\vec{P}_1,\vec{P}_2,\vec{P}_4)R_2(\vec{P}_3,\vec{P}_5)\\
&+R_3(\vec{P}_1,\vec{P}_2,\vec{P}_5)R_2(\vec{P}_3,\vec{P}_4)+R_3(\vec{P}_1,\vec{P}_3,\vec{P}_4)R_2(\vec{P}_2,\vec{P}_5)+R_3(\vec{P}_1,\vec{P}_3,\vec{P}_5)R_2(\vec{P}_2,\vec{P}_4)\\
&+R_3(\vec{P}_1,\vec{P}_4,\vec{P}_5)R_2(\vec{P}_2,\vec{P}_3)+R_3(\vec{P}_2,\vec{P}_3,\vec{P}_4)R_2(\vec{P}_1,\vec{P}_5)+R_3(\vec{P}_2,\vec{P}_3,\vec{P}_5)R_2(\vec{P}_1,\vec{P}_4)\\
&+R_3(\vec{P}_2,\vec{P}_4,\vec{P}_5)R_2(\vec{P}_1,\vec{P}_3)+R_3(\vec{P}_3,\vec{P}_4,\vec{P}_5)R_2(\vec{P}_1,\vec{P}_2)]+2[R_2(\vec{P}_1,\vec{P}_2)R_2(\vec{P}_3,\vec{P}_4)\\
&+R_2(\vec{P}_1,\vec{P}_2)R_2(\vec{P}_3,\vec{P}_5)+R_2(\vec{P}_1,\vec{P}_2)R_2(\vec{P}_4,\vec{P}_5)+R_2(\vec{P}_1,\vec{P}_3)R_2(\vec{P}_2,\vec{P}_4)\\
&+R_2(\vec{P}_1,\vec{P}_3)R_2(\vec{P}_2,\vec{P}_5)+R_2(\vec{P}_1,\vec{P}_3)R_2(\vec{P}_4,\vec{P}_5)+R_2(\vec{P}_1,\vec{P}_4)R_2(\vec{P}_2,\vec{P}_3)\\
&+R_2(\vec{P}_1,\vec{P}_4)R_2(\vec{P}_2,\vec{P}_5)+R_2(\vec{P}_1,\vec{P}_4)R_2(\vec{P}_3,\vec{P}_5)+R_2(\vec{P}_1,\vec{P}_5)R_2(\vec{P}_2,\vec{P}_3)\\
&+R_2(\vec{P}_1,\vec{P}_5)R_2(\vec{P}_2,\vec{P}_4)+R_2(\vec{P}_1,\vec{P}_5)R_2(\vec{P}_3,\vec{P}_4)+R_2(\vec{P}_2,\vec{P}_3)R_2(\vec{P}_4,\vec{P}_5)\\
&+R_2(\vec{P}_2,\vec{P}_4)R_2(\vec{P}_3,\vec{P}_5)+R_2(\vec{P}_2,\vec{P}_5)R_2(\vec{P}_3,\vec{P}_4)]-6[R_2(\vec{P}_1,\vec{P}_2)+R_2(\vec{P}_1,\vec{P}_3)+R_2(\vec{P}_1,\vec{P}_4)\\
&+R_2(\vec{P}_1,\vec{P}_5)+R_2(\vec{P}_2,\vec{P}_3)+R_2(\vec{P}_2,\vec{P}_4)+R_2(\vec{P}_2,\vec{P}_5)+R_2(\vec{P}_3,\vec{P}_4)+R_2(\vec{P}_3,\vec{P}_5)+R_2(\vec{P}_4,\vec{P}_5)]\\
=&R_5(\vec{P}_1,\vec{P}_2,\vec{P}_3,\vec{P}_4,\vec{P}_5)-1-[k_4(\vec{P}_1,\vec{P}_2,\vec{P}_3,\vec{P}_4)+k_4(\vec{P}_1,\vec{P}_2,\vec{P}_3,\vec{P}_5)+k_4(\vec{P}_1,\vec{P}_2,\vec{P}_4,\vec{P}_5)\\
&+k_4(\vec{P}_1,\vec{P}_3,\vec{P}_4,\vec{P}_5)+k_4(\vec{P}_2,\vec{P}_3,\vec{P}_4,\vec{P}_5)]-[k_3(\vec{P}_1,\vec{P}_2,\vec{P}_3)+k_3(\vec{P}_1,\vec{P}_2,\vec{P}_4)+k_3(\vec{P}_1,\vec{P}_2,\vec{P}_5)\\
&+k_3(\vec{P}_1,\vec{P}_3,\vec{P}_4)+k_3(\vec{P}_1,\vec{P}_3,\vec{P}_5)+k_3(\vec{P}_1,\vec{P}_4,\vec{P}_5)+k_3(\vec{P}_2,\vec{P}_3,\vec{P}_4)+k_3(\vec{P}_2,\vec{P}_3,\vec{P}_5)\\
&+k_3(\vec{P}_2,\vec{P}_4,\vec{P}_5)+k_3(\vec{P}_3,\vec{P}_4,\vec{P}_5)]-[k_3(\vec{P}_1,\vec{P}_2,\vec{P}_3)k_2(\vec{P}_4,\vec{P}_5)+k_3(\vec{P}_1,\vec{P}_2,\vec{P}_4)k_2(\vec{P}_3,\vec{P}_5)\\
&+k_3(\vec{P}_1,\vec{P}_2,\vec{P}_5)k_2(\vec{P}_3,\vec{P}_4)+k_3(\vec{P}_1,\vec{P}_3,\vec{P}_4)k_2(\vec{P}_2,\vec{P}_5)+k_3(\vec{P}_1,\vec{P}_3,\vec{P}_5)k_2(\vec{P}_2,\vec{P}_4)\\
&+k_3(\vec{P}_1,\vec{P}_4,\vec{P}_5)k_2(\vec{P}_2,\vec{P}_3)+k_3(\vec{P}_2,\vec{P}_3,\vec{P}_4)k_2(\vec{P}_1,\vec{P}_5)+k_3(\vec{P}_2,\vec{P}_3,\vec{P}_5)k_2(\vec{P}_1,\vec{P}_4)\\
&+k_3(\vec{P}_2,\vec{P}_4,\vec{P}_5)k_2(\vec{P}_1,\vec{P}_3)+k_3(\vec{P}_3,\vec{P}_4,\vec{P}_5)k_2(\vec{P}_1,\vec{P}_2)]-[k_2(\vec{P}_1,\vec{P}_2)k_2(\vec{P}_3,\vec{P}_4)\\
&+k_2(\vec{P}_1,\vec{P}_2)k_2(\vec{P}_3,\vec{P}_5)+k_2(\vec{P}_1,\vec{P}_2)k_2(\vec{P}_4,\vec{P}_5)+k_2(\vec{P}_1,\vec{P}_3)k_2(\vec{P}_2,\vec{P}_4)+k_2(\vec{P}_1,\vec{P}_3)k_2(\vec{P}_2,\vec{P}_5)\\
&+k_2(\vec{P}_1,\vec{P}_3)k_2(\vec{P}_4,\vec{P}_5)+k_2(\vec{P}_1,\vec{P}_4)k_2(\vec{P}_2,\vec{P}_3)+k_2(\vec{P}_1,\vec{P}_4)k_2(\vec{P}_2,\vec{P}_5)+k_2(\vec{P}_1,\vec{P}_4)k_2(\vec{P}_3,\vec{P}_5)\\
&+k_2(\vec{P}_1,\vec{P}_5)k_2(\vec{P}_2,\vec{P}_3)+k_2(\vec{P}_1,\vec{P}_5)k_2(\vec{P}_2,\vec{P}_4)+k_2(\vec{P}_1,\vec{P}_5)k_2(\vec{P}_3,\vec{P}_4)+k_2(\vec{P}_2,\vec{P}_3)k_2(\vec{P}_4,\vec{P}_5)\\
&+k_2(\vec{P}_2,\vec{P}_4)k_2(\vec{P}_3,\vec{P}_5)+k_2(\vec{P}_2,\vec{P}_5)k_2(\vec{P}_3,\vec{P}_4)]-[k_2(\vec{P}_1,\vec{P}_2)+k_2(\vec{P}_1,\vec{P}_3)+k_2(\vec{P}_1,\vec{P}_4)\\
&+k_2(\vec{P}_1,\vec{P}_5)+k_2(\vec{P}_2,\vec{P}_3)+k_2(\vec{P}_2,\vec{P}_4)+k_2(\vec{P}_2,\vec{P}_5)+k_2(\vec{P}_3,\vec{P}_4)+k_2(\vec{P}_3,\vec{P}_5)+k_2(\vec{P}_4,\vec{P}_5)].
\end{aligned} \tag{16}$$

III. DISCUSSION

A. Normalized cumulants with coherence

By substituting the correlation functions given in Eqs. (2), (5), (6), and (7) the normalized cumulants can be written directly in terms of the Bose-Einstein amplitudes b_{ij}. In performing this substitution, we will separate the normalized cumulant k_n of order n into a term $k_n^{(I)}$ which arises from the incoherent component of the source and a term $k_n^{(C-I)}$ which arises from the coherent-incoherent interference cross term. Thus, $k_n = k_n^{(I)} + k_n^{(C-I)}$.

For the two-particle normalized cumulants ($n=2$) we then have

$$k_2^{(I)} = \epsilon^2 b_{12}^2, \tag{17}$$

$$k_2^{(C-I)} = 2\ \phi\epsilon b_{12}. \tag{18}$$

For the three-particle normalized cumulants ($n=3$) we have

$$k_3^{(I)} = 2\ \epsilon^3 b_{12} b_{13} b_{23}, \tag{19}$$

$$k_3^{(C-I)} = 2\ \phi\epsilon^2 (b_{12}b_{13} + b_{12}b_{23} + b_{13}b_{23}). \tag{20}$$

For the four-particle normalized cumulants ($n=4$) we have

$$k_4^{(I)} = 2\ \epsilon^4 (b_{13}b_{14}b_{23}b_{24} + b_{12}b_{14}b_{23}b_{34} + b_{12}b_{13}b_{24}b_{34}), \tag{21}$$

$$\begin{aligned} k_4^{(C-I)} = 2\ \phi\epsilon^3 [& b_{12}b_{14}b_{23} + b_{13}b_{14}b_{23} + b_{12}b_{13}b_{24} + b_{13}b_{24}b_{34} + b_{13}b_{14}b_{24} + b_{13}b_{23}b_{24} + b_{14}b_{23}b_{24} + b_{12}b_{13}b_{34} + b_{12}b_{14}b_{34} \\ & + b_{12}b_{23}b_{34} + b_{14}b_{23}b_{34} + b_{12}b_{24}b_{34}]. \end{aligned} \tag{22}$$

For the five-particle normalized cumulants ($n=5$) we have

$$\begin{aligned} k_5^{(I)} = 2\ \epsilon^5 (& b_{14}b_{15}b_{23}b_{25}b_{34} + b_{13}b_{15}b_{24}b_{25}b_{34} + b_{14}b_{15}b_{23}b_{24}b_{35} + b_{13}b_{14}b_{24}b_{25}b_{35} + b_{12}b_{15}b_{24}b_{34}b_{35} + b_{12}b_{14}b_{25}b_{34}b_{35} \\ & + b_{13}b_{15}b_{23}b_{24}b_{45} + b_{13}b_{14}b_{23}b_{25}b_{45} + b_{12}b_{15}b_{23}b_{34}b_{45} + b_{12}b_{13}b_{25}b_{34}b_{45} + b_{12}b_{14}b_{23}b_{35}b_{45} + b_{12}b_{13}b_{24}b_{35}b_{45}), \end{aligned} \tag{23}$$

$$\begin{aligned} k_5^{(C-I)} = 2\ \phi\epsilon^4 (& b_{13}b_{15}b_{23}b_{24} + b_{14}b_{15}b_{23}b_{24} + b_{13}b_{14}b_{23}b_{25} + b_{14}b_{15}b_{23}b_{25} + b_{13}b_{14}b_{24}b_{25} + b_{13}b_{15}b_{24}b_{25} + b_{12}b_{15}b_{23}b_{34} \\ & + b_{14}b_{15}b_{23}b_{34} + b_{12}b_{15}b_{24}b_{34} + b_{13}b_{15}b_{24}b_{34} + b_{12}b_{13}b_{25}b_{34} + b_{12}b_{14}b_{25}b_{34} + b_{13}b_{15}b_{25}b_{34} + b_{14}b_{15}b_{25}b_{34} \\ & + b_{14}b_{23}b_{25}b_{34} + b_{15}b_{23}b_{25}b_{34} + b_{13}b_{24}b_{25}b_{34} + b_{15}b_{24}b_{25}b_{34} + b_{12}b_{14}b_{23}b_{35} + b_{14}b_{15}b_{23}b_{35} + b_{12}b_{13}b_{24}b_{35} \\ & + b_{13}b_{14}b_{24}b_{35} + b_{12}b_{15}b_{24}b_{35} + b_{14}b_{15}b_{24}b_{35} + b_{14}b_{23}b_{24}b_{35} + b_{15}b_{23}b_{24}b_{35} + b_{12}b_{14}b_{25}b_{35} + b_{13}b_{14}b_{25}b_{35} \\ & + b_{13}b_{24}b_{25}b_{35} + b_{14}b_{24}b_{25}b_{35} + b_{12}b_{14}b_{34}b_{35} + b_{12}b_{15}b_{34}b_{35} + b_{12}b_{24}b_{34}b_{35} + b_{15}b_{24}b_{34}b_{35} + b_{12}b_{25}b_{34}b_{35} \\ & + b_{14}b_{25}b_{34}b_{35} + b_{12}b_{14}b_{23}b_{45} + b_{13}b_{14}b_{23}b_{45} + b_{12}b_{15}b_{23}b_{45} + b_{13}b_{15}b_{23}b_{45} + b_{12}b_{13}b_{24}b_{45} + b_{13}b_{15}b_{24}b_{45} \\ & + b_{13}b_{23}b_{24}b_{45} + b_{15}b_{23}b_{24}b_{45} + b_{12}b_{13}b_{25}b_{45} + b_{13}b_{14}b_{25}b_{45} + b_{13}b_{23}b_{25}b_{45} + b_{14}b_{23}b_{25}b_{45} + b_{12}b_{13}b_{34}b_{45} \\ & + b_{12}b_{15}b_{34}b_{45} + b_{12}b_{23}b_{34}b_{45} + b_{15}b_{23}b_{34}b_{45} + b_{12}b_{25}b_{34}b_{45} + b_{13}b_{25}b_{34}b_{45} + b_{12}b_{13}b_{35}b_{45} + b_{12}b_{14}b_{35}b_{45} \\ & + b_{12}b_{23}b_{35}b_{45} + b_{14}b_{23}b_{35}b_{45} + b_{12}b_{24}b_{35}b_{45} + b_{13}b_{24}b_{35}b_{45}). \end{aligned} \tag{24}$$

In the normalized cumulants above, we note that the incoherent contributions given in Eqs. (17), (19), (21), and (23) are what has been called the "true" multiparticle correlation [2]. However, the coherent-incoherent cross terms given in Eqs. (18), (20), (22), and (24) may also represent significant contributions to the normalized cumulants when coherence is present, and these are composed of correlation terms which do not reflect the full multiparticle correlation of order n, in that each term is missing one "link" between a pair of pions in the correlation.

It should be noted that in the the full normalized cumulants given above, when we remove particle n to infinity ($q_{in} \to \infty$, with $i=1,\ldots,n-1$) so that $b_{in}=0$, both k_4^I and k_4^{C-I} become zero. Thus, in some sense the full normalized cumulant, even in the presence of coherence, reflects the multiparticle correlation because all terms in Eqs. (17)–(24) depend on correlations with all particles present. This is not what is normally meant by the "true" multiparticle correlation, however.

We note a remarkable correspondence between the above relations and the linked-pair approximation [3]. The incoherent parts of the cumulants given in Eqs. (17), (19), (21), and (23) above are superpositions of all the ways in which the correlated particles can be linked to form a closed ring. The effect of coherence is to delete one of these links, and the coherent-incoherent cross terms of Eqs. (18), (20), (22), and (24). These are the same ring correlations with one link in the ring deleted, and they therefore form on open chain of linked pairs. This open chain has the same correlation topology as that used in the linked-pair approximation. In particular, we note that the linked-pair approximation expressions for high order cumulants given in Eq. (25) of Ref. [3] for $n=3$–5 are the same to within a normalization factor as our Eqs. (20), (22), and (24). Thus, the linked-pair approximation is *de facto* equivalent to the assumption of dominant coherence in the correlation.

B. Evaluation of correlation functions and normalized cumulants

Let us first consider how the relations presented above can be compared with experimental data. There is a fundamental problem in histogramming and fitting multiparticle correlation functions and cumulants because there is a very large number of independent momentum variables. Thus, there is no one ''correct'' way of plotting, histogramming, or fitting multiparticle correlations with $n>2$. One of us (J.G.C.) is co-author of a recent publication [10] which provides a procedure for avoiding any binning of the data and using the maximum likelihood method to directly fit a given data set with a ''hypothesis'' probability density provided by multiparticle correlation functions like those given in Eqs. (2), (5), (6), and (7) used with a model [like the Gaussian model used in deriving Eq. (3)] which relates the $b_{i,j}$ to momentum variables.

An alternate approach was presented by one of us (J.G.C.) in an earlier publication [4]. There it was shown that any multiparticle system can be represented in terms of three ''coalescence variables,'' momentum-related quantities giving the increase in the invariant mass of the system over its at-rest value due to motion of the component particles in the transverse, longitudinal, and radial directions, respectively. A theoretical correlation model was used to show that these variables could be used to analyze multiparticle correlations by binning data with these variables and then fitting the resulting three-dimensional histograms with a theoretical model that is represented in terms of the same variables, while averaging over all other momentum variables. The multiparticle correlation function given in Eqs. (2), (5), (6), and (7), with an appropriate model for the b_{ij}, would be used in this way.

In a recent unpublished Ph.D. thesis, Brinkmann [13] used momentum variables similar to those of Ref. [4] to investigate Bose-Einstein correlations in data from experiment NA35 for multiparticle groups up to $n=5$. These experimental multiparticle correlation functions were not, however, fitted with the correlation formalism presented here (the formalism had not yet been derived) and nor were coherence and contamination explicitly considered.

For the purposes of the present work, let us consider the very simple illustrative example of the correlations of four particles under the restrictive simplifying assumptions that $\kappa=0$, i.e., no contamination, that $b_{ij}=\exp[-(q_{ij}r)^2]$, i.e., Gaussian Bose-Einstein amplitudes, and that $q_{ij}=q$ for all combinations of i and j, i.e., the correlated particles lie at the corners of a regular tetrahedron in momentum space. The latter assumption, which has been widely used in the literature of Bose-Einstein correlations, is, of course, very restrictive, but allows us to investigate the effects of coherence and contamination. We will consider the cases with $r = 6$ fm with $\phi=1/8$ (small coherence) and $\phi=1/2$ (medium coherence).

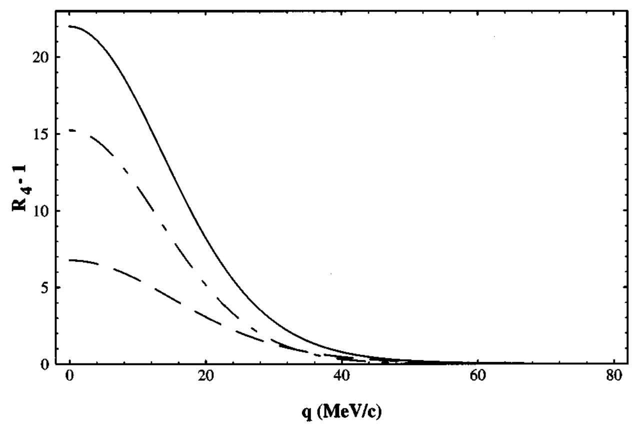

FIG. 1. Modified four-particle correlation function R_4-1 vs momentum difference q (see text) for the case $\epsilon = 7/8$. The solid curve is the total correlation, the dash-dotted curve is the incoherent contribution, and the dashed curve is the contribution of the coherent-incoherent cross term.

Figure 1 shows the modified four-particle correlation function R_4-1 for the case $\phi = 1/8$, plotted against the momentum difference q in units of MeV/c. Three curves are shown, the total correlation function and the separate contributions from the incoherent term and the coherent-incoherent cross term. We note that at the origin ($q = 0$) the three functions have values $R_4(0)=22.0$, $R_4^I(0)=15.2$, and $R_4^{C-I}(0)=6.77$; i.e., the incoherent contribution exceeds the coherent-incoherent cross term by more than a factor of 2. Note also that at about $q = 34$ MeV/c the two contributions become equal because of the broader width of the cross term.

Figure 2 shows a similar plot of R_4-1 for the case $\phi =$

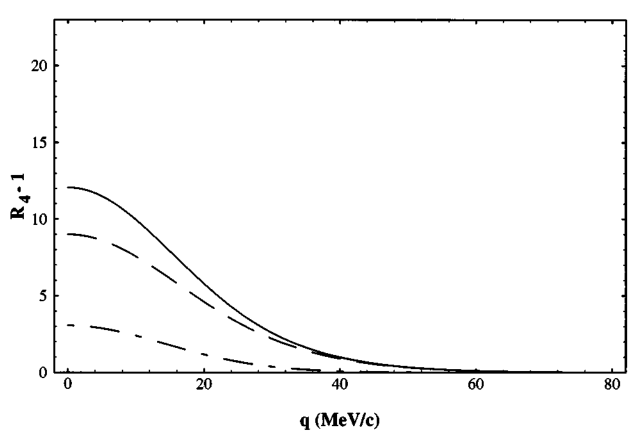

FIG. 2. Modified four-particle correlation function R_4-1 vs momentum difference q (see text) for the case $\epsilon = 1/2$. The solid curve is the total correlation, the dash-dotted curve is the incoherent contribution, and the dashed curve is the contribution of the coherent-incoherent cross term.

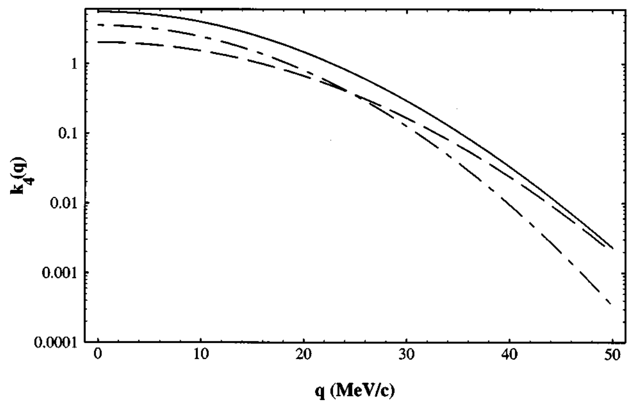

FIG. 3. Normalized fourth-order cumulant k_4 vs momentum difference q (see text) for the case ϵ = 7/8. The solid curve is the total cumulant, the dash-dotted curve is the incoherent contribution, and the dashed curve is the contribution of the coherent-incoherent cross term.

1/2. Here the coherent-incoherent cross term dominates, and at the origin $(q=0)$ the three functions plotted have values $R_4(0)=12.1$, $R_4^I(0)=3.06$, and $R_4^{C-I}(0)=9.0$; i.e., the coherent-incoherent cross term exceeds the incoherent contribution by almost a factor of 3.

Figure 3 shows the four-particle normalized cumulant k_4 for the case ϕ = 1/8, again plotted against the momentum difference q in units of MeV/c. Three curves are shown, the total correlation function and the separate contributions from the incoherent term and the coherent-incoherent cross term. We note that at the origin the three functions have values $k_4(0)=5.53$, $k_4^I(0)=3.52$, and $k_4^{C-I}(0)=2.01$; i.e., the incoherent contribution exceeds the coherent-incoherent cross term by a factor of about 1.7. Note also that at about q = 25 MeV/c the two contributions become equal because of the broader width of the cross term.

Figure 4 shows a similar plot of k_4 for the case ϕ = 1/2. Here the coherent-incoherent cross term dominates, and at the origin $(q=0)$ the three functions plotted have values $k_4(0)=1.88$, $k_4^I(0)=0.375$, and $k_4^{C-I}(0)=1.5$; i.e., the coherent-incoherent cross term exceeds the incoherent contribution by a factor of 4.

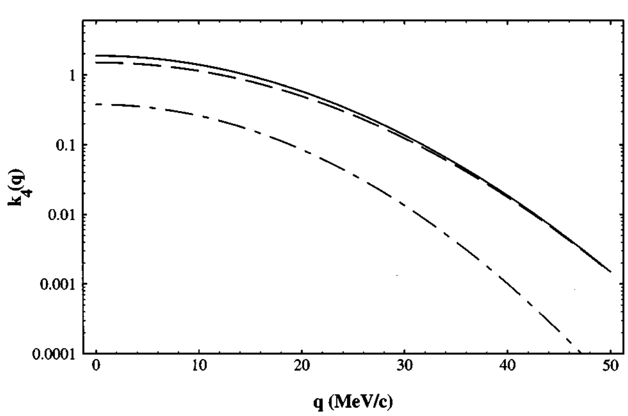

FIG. 4. Normalized fourth-order cumulant k_4 vs momentum difference q (see text) for the case ϵ = 1/2. The solid curve is the total cumulant, the dash-dotted curve is the incoherent contribution, and the dashed curve is the contribution of the coherent-incoherent cross term.

C. Effect of coherence and contamination on intercepts

In experimental measurements of Bose-Einstein correlation functions in ultrarelativistic heavy ion experiments, it is usually found that the correlation functions R_n, after suitable corrections for Coulomb effects and contaminations from particles that are not pions, do not reach the "intercept" values at $Q=0$ that would be expected if the emission process was purely incoherent [11–13]. It is an open question whether this observation of a reduced correlation is related to the coherent effects discussed above, to resonance effects, or to problems with the correction procedures used.

Here we point out that the relations presented above in Eqs. (2)–(24) provide systematic predictions of the intercept values of $R_n(0)$ and $k_n(0)$ as a function of n, the order of the correlation. For $n=$ 2–5 the intercepts were calculated from the above relations. These intercepts are given by the following expressions:

$$R_2(0)=1+\epsilon^2+2\,\phi\epsilon, \tag{25}$$

$$R_3(0)=1+3\epsilon^2+2\epsilon^3+6\phi(\epsilon+\epsilon^2), \tag{26}$$

$$R_4(0)=1+6\epsilon^2+8\epsilon^3+9\,\epsilon^4+\phi(12\,\epsilon+24\,\epsilon^2+36\,\epsilon^3)+12\,\phi^2\epsilon^2, \tag{27}$$

$$R_5(0)=1+10\epsilon^2+20\epsilon^3+45\epsilon^4+44\epsilon^5+\phi(20\epsilon+60\epsilon^2+180\epsilon^3+220\epsilon^4)+\phi^2(60\,\epsilon^2+120\epsilon^3), \tag{28}$$

$$k_2(0)=\epsilon^2+2\phi\epsilon, \tag{29}$$

$$k_3(0)=2\epsilon^3+6\phi\epsilon^2, \tag{30}$$

$$k_4(0)=6\epsilon^4+24\phi\epsilon^3, \tag{31}$$

$$k_5(0)=24\epsilon^5+120\phi\epsilon^4. \tag{32}$$

Let us define a reduction factor variable $\delta\equiv 2-R_2(0)=1-k_2(0)$; i.e., δ is for $n=2$ the amount by which the intercept values of the correlation function R_2 and the normalized cumulant k_2 are reduced. With $\kappa=0$ and $\phi=(1-\epsilon)$, we then have $\epsilon=1-\sqrt{\delta}$, and with $\phi=0$ and $\kappa=(1-\epsilon)$ we have $\epsilon=\sqrt{1-\delta}$. We can therefore plot the intercept values of R_n and k_n against δ by evaluating ϵ as an intermediate step.

Figure 5 shows the effect of increasing coherence on the intercepts of the correlation functions $R_n(0)$ for $n=2$–5. With $\kappa=0$ and $\phi=1-\epsilon$, we plot the fractional decrease of the correlation functions $R_n(0)/R_n(\mathrm{max})$ against δ $(0\leqslant\delta\leqslant 1)$. For this case the coherence fraction $\phi=\sqrt{\delta}$.

Figure 6 shows the effect of increasing coherence on the intercepts of the normalized cumulants $k_n(0)$ for $n=2$–5. With $\kappa=0$ and $\phi=1-\epsilon$, we plot the fractional decrease of the correlation functions $R_n(0)/R_n(\mathrm{max})$ against δ $(0\leqslant\delta\leqslant 1)$. Here again the coherence fraction $\phi=\sqrt{\delta}$.

Figure 7 shows the effect of increasing contamination on the intercepts of the correlation functions $R_n(0)$ for $n=2$–5.

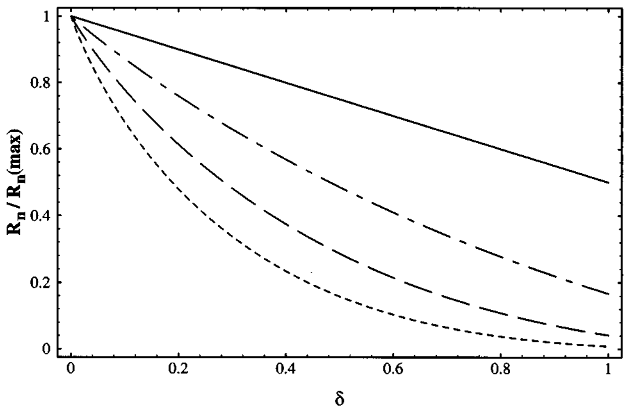

FIG. 5. The effect of increasing coherence on the intercepts of correlations functions $R_n(0;\epsilon)$ for $n=2$–5. Here $\delta=R_2(0;\epsilon)-1$. Assuming that this reduction is due to coherence and that $\kappa=0$, we plot $R_n(0,\epsilon)/R_n(0,\epsilon=1)$ against δ for $n=2$–5. The correlation intercepts for $n = 2$, 3, 4, and 5 are shown by the solid, dot-dashed, long dashed, and short dashed curves, respectively.

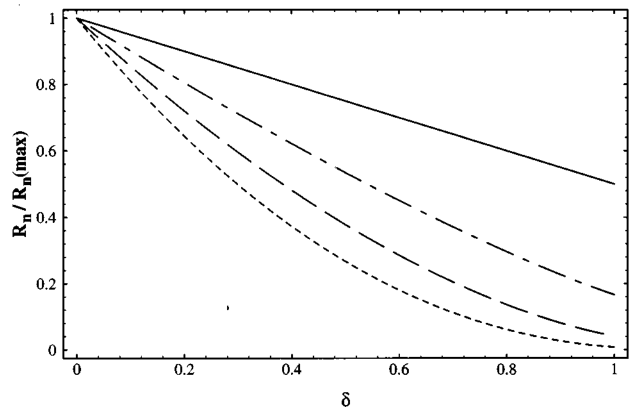

FIG. 7. The effect of increasing contamination on the intercepts of correlations functions $R_n(0;\epsilon)$ for $n=2$–5. Here $\delta=R_2(0;\epsilon)-1$. Assuming that this reduction is due to contamination and that $\psi=0$, we plot $R_n(0,\epsilon)/R_n(0,\epsilon=1)$ against δ for $n=$ 2–5. The correlation intercepts for $n = 2$, 3, 4, and 5 are shown by the solid, dot-dashed, long dashed, and short dashed curves, respectively.

With $\phi=0$ and $\kappa=1-\epsilon$, we plot the fractional decrease of the correlation functions $R_n(0)/R_n(\mathrm{max})$ against δ $(0\leqslant\delta\leqslant1)$. For this case the contamination fraction $\kappa=1-\sqrt{1-\delta}$.

Figure 8 shows the effect of increasing contamination on the intercepts of the normalized cumulants $k_n(0)$for $n=2$–5. With $\phi=0$ and $\kappa=1-\epsilon$, we plot the fractional decrease of the correlation functions $R_n(0)/R_n(\mathrm{max})$ against δ $(0\leqslant\delta\leqslant1)$. Here again the contamination fraction $\kappa=1-\sqrt{1-\delta}$.

Comparing Figs. 6 and 8 indicates that for a given reduction in the k_2 intercept, the reduction due to coherence in the intercepts of the higher normalized cumulants ($n = 3$–5) is much stronger than the analogous reductions produced by contamination. We suggest that this difference in functional dependence might be used to distinguish the two effects in experimental data.

IV. CONCLUSION

We have considered the effect of coherence on Bose-Einstein correlation functions and normalized cumulants of orders 2, 3, 4, and 5. We find that the presence of coherence produces a coherent-incoherent interference amplitude which adds additional terms to the correlation functions and cumulants, and in particular means that the normalized cumulant no longer isolates the "true" multiparticle correlation. We have shown the dependence of the relative sizes of the coherent and incoherent contributions on the fractional incoherence. Finally, we have considered the behavior of the correlation functions and normalized cumulants in the region of $Q=0$. We chose levels of coherence and contamination which had the same reduction effect on the $n=2$ correlation and found that for the normalized cumulants of high order, the reduction due to coherence was much stronger than an analogous reduction due to contamination.

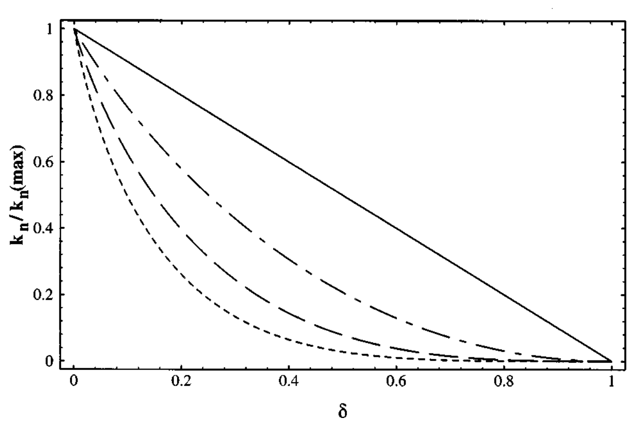

FIG. 6. The effect of increasing coherence on the intercepts of normalized cumulants $k_n(0;\epsilon)$ for $n=2$–5. Here $\delta=k_2(0;\epsilon)$. Assuming that this reduction is due to coherence and that $\kappa=0$, we plot $k_n(0,\epsilon)/k_n(0,\epsilon=1)$ against δ for $n=2$–5. The normalized cumulants for $n = 2$, 3, 4, and 5 are shown by the solid, dot-dashed, long dashed, and short dashed curves, respectively.

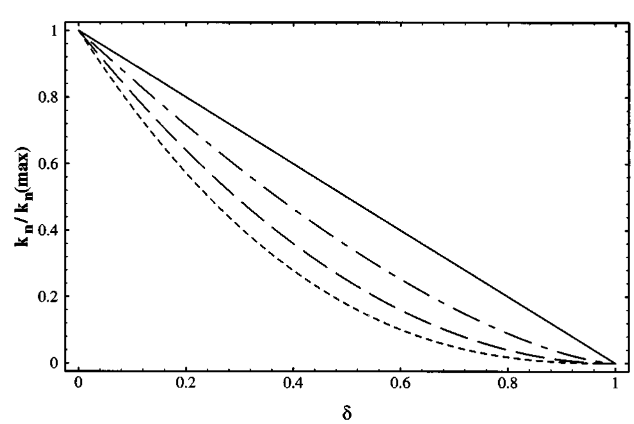

FIG. 8. The effect of increasing contamination on the intercepts of normalized cumulants $k_n(0;\epsilon)$ for $n=2$–5. Here $\delta=k_2(0;\epsilon)$. Assuming that this reduction is due to contamination and that $\psi=0$, we plot $k_n(0,\epsilon)/k_n(0,\epsilon=1)$ against δ for n $=2$ to 5. The normalized cumulants for $n=2$, 3, 4, and 5 are shown by the solid, dot-dashed, long dashed, and short dashed curves, respectively.

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