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Analysis of multiparticle Bose-Einstein correlations in ultrarelativistic heavy ion collisions

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We introduce the *coalescence variables*, a set of three boost-invariant kinematic quantities that may be used in analyzing n-particle correlations. These variables characterize the invariant mass of an n-particle system, and in three directions separate the timelike and spacelike characteristics of the source. The analytic Kolehmanien-Gyulassy model is generalized to give two-, three-, and four-particle correlation functions, with coherence and Coulomb corrections applied to the basic formalism. We demonstrate the relation of the coalescence variables to the radius and duration of the source, and find that for sufficiently large transverse radii, Coulomb effects can suppress the structure of the Hanbury-Brown–Twiss correlations so that no significant information on source size can be obtained.

I. INTRODUCTION

In ultrarelativistic collisions between heavy ions, the number of like-sign pions produced in a single collision is expected to be on the order of 10^3. Because of this very large multiplicity, Hanbury-Brown–Twiss correlations between pions may offer a powerful probe for the investigation of such collisions on an event-by-event basis. The numerical factors implicit in multiparticle Bose-Einstein interferometry imply a strong tendency for pions to cluster or "coalesce" in the same region of momentum space due to their mutual Bose-Einstein reinforcement. Therefore, interferometry using pions clustered in momentum space may offer an important analysis tool. Indeed, it has been suggested[1,2] that pion "speckle interferometry," i.e., the high-order correlations of pion clusters or "speckles," might be used to extract detailed information on the size, shape, time duration, and eccentricity of the source of pion emission.

A fundamental problem encountered in interferometry using correlations between n particles, where n is larger than 2, is finding a compact set of independent Lorentz-invariant kinematic basis variables for presenting and analyzing the correlations. The relative vector momenta of n particles require 3^{n-1} independent variables for complete specification. This number of parameters is far too large and too inter-related for meaningful analysis. For example, analysis using the relative vector momenta in the correlation of a five-pion system would require 27 independent momentum variables.

Goldhaber[3] was able to compare the correlations of two and three pion systems by plotting both correlation distributions against a variable which he called Q^2, the mean-square deviation of the invariant mass of the n-pion system from its minimum possible value, i.e., $Q^2 = E^\mu E_\mu - (n\,\mu_\pi)^2$. Liu *et al.*[4] have employed this Q^2 variable in analyzing two- and three-pion correlations in Ar + Pb and Ar + KCl heavy ion collisions. For the purposes of the present discussion, we will refer to Q as the overall *coalescence* variable, since it goes to zero when a system of pions has coalesced to occupy a minimum volume of momentum space. The use of Q^2 and similar variables in the analysis of correlations of relativistic particles has been criticized[5,6] because, although it is a Lorentz-invariant quantity, it mixes the timelike and spacelike characteristics of the source. In central collisions of ultrarelativistic heavy ions the time and space (or longitudinal and transverse) source characteristics provide independent information about the collision, and it is important to keep these separated in the analysis of multiparticle correlations.

The invariant mass of a two-pion system can exceed the minimum value of $2\,\mu_\pi$ only if the two pions have nonzero relative momentum. Consider the distribution of emitted pions in a spherical coordinate system (r, θ, ϕ) where the beam direction is the z or longitudinal axis defined by $\theta = \pi/2$ and the locus $\theta = 0$ is the transverse (equatorial) plane. Thus, θ has the range $-\pi/2 \leq \theta \leq \pi/2$. The relative momentum between a pair of emitted pions can have longitudinal (θ), transverse (ϕ), and radial (r) components. In the context of Hanbury-Brown–Twiss correlations, these three momentum differences sample separate geometrical aspects of the pion source and should, if possible, be investigated separately. In what follows, we propose a decomposition of the Goldhaber coalescence variable into longitudinal, transverse, and radial coalescence components, thereby preserving this distinction. We then apply this analysis technique to a comparison of two-, three-, and four-pion systems using a generalization of the analytic model of Kolehmainen and Gyulassy.[7,8]

II. COALESCENCE IN THE 2-PARTICLE SYSTEM

We will use the following notation. The rest mass of the ith particle is μ_π, i.e., the pion mass. Its total energy is w_i, its vector momentum is $\mathbf{P}_i$, its transverse momentum is p_i, its transverse mass is $m_i = \sqrt{p_i^2 + \mu_\pi^2}$, its longitudinal momentum is $q_i = \sqrt{w_i^2 - m_i^2}$, its azimuthal angle in the transverse plane is ϕ_i, its polar angle forward or backward of the transverse plane is θ_i, and its rapidity is $y_i = \frac{1}{2}\ln[(w_i + q_i)/(w_i - q_i)] \approx \ln[(1 + \sin\theta_i)/\cos\theta_i]$. We will also employ the useful rapidity relations $w_i = m_i\cosh(y_i)$ and $q_i = m_i\sinh(y_i)$. We will take $c = 1$ and express all of the above quantities except angle and rapidity in energy units, usually MeV.

For a two-particle system, the Goldhaber coalescence variable is given by

$$\begin{aligned} Q_2^2 &= (w_1 + w_2)^2 - (\mathbf{P}_1 + \mathbf{P}_2)^2 - (2\mu_\pi)^2 \\ &= (w_1 + w_2)^2 - (p_1\cos\phi_1 + p_2\cos\phi_2)^2 \\ &\quad -(p_1\sin\phi_1 + p_2\sin\phi_2)^2 - (q_1 + q_2)^2 - (2\mu_\pi)^2 . \end{aligned} \tag{1}$$

With suitable algebraic manipulation, this can be reduced to the following form:

$$Q_2^2 = 2\left\{ m_1 m_2 \left[1 + 2\sinh^2\left(\frac{y_1 - y_2}{2}\right)\right] - p_1 p_2 \left[1 - 2\sin^2\left(\frac{\phi_1 - \phi_2}{2}\right)\right] - \mu_\pi^2 \right\} . \tag{2}$$

We make the following definitions of dimensionless coalescence variables:

$$C_{2L}(1,2) = \frac{1}{\mu_\pi}\sqrt{2m_1m_2}\ \sinh\left|\frac{y_1 - y_2}{2}\right| , \tag{3}$$

$$C_{2T}(1,2) = \frac{1}{\mu_\pi}\sqrt{2p_1p_2}\ \sin\left|\frac{\phi_1 - \phi_2}{2}\right| , \tag{4}$$

$$\begin{aligned} C_{2R}(1,2) &= \frac{1}{\mu_\pi}\sqrt{m_1m_2 - p_1p_2 - \mu_\pi^2} \\ &= \frac{1}{\mu_\pi}\sqrt{[(p_1 - p_2)^2 - (m_1 - m_2)^2]/2} \\ &= \frac{1}{\mu_\pi}\sqrt{[(m_1 + m_2)^2 - (p_1 + p_2)^2 - (2\mu_\pi)^2]/2} . \end{aligned} \tag{5}$$

Here C_{2L} is the two-particle *longitudinal coalescence*, C_{2T} is the two-particle *transverse coalescence*, and C_{2R} is the two-particle *radial coalescence*. They are dimensionless variables that specify the invariant mass deviation relative to μ_π, the pion mass.

Note that C_{2L}=0 when y_1=y_2, that C_{2T}=0 when ϕ_1=ϕ_2, and that C_{2R}=0 when either m_1=m_2 or p_1=p_2 (since each equality implies the other). The overall two particle coalescence Q_2 is given by the relation $Q_2^2 = 2\mu_\pi^2(C_{2L}^2 + C_{2T}^2 + C_{2R}^2)$, i.e., the three coalescence components add in quadrature, as would be expected of orthogonal coordinates.

III. COALESCENCE IN THE n-PARTICLE SYSTEM

Now let us consider the case of n correlated particles, where n>2. The n-particle coalescence is given by

$$Q_n^2 = \left(\sum_{i=1}^{n} w_i\right)^2 - \left(\sum_{i=1}^{n} \mathbf{P}_i\right)^2 - \left(\sum_{i=1}^{n} \mu_\pi\right)^2 , \tag{6}$$

which with suitable manipulation becomes

$$\begin{aligned} Q_n^2 = &\sum_{i\neq j}^{n} m_i m_j \left[1 + 2\sinh^2\left(\frac{y_i - y_j}{2}\right)\right] \\ &- \sum_{i\neq j}^{n} p_i p_j \left[1 - 2\sin^2\left(\frac{\phi_i - \phi_j}{2}\right)\right] - \sum_{i\neq j}^{n} \mu_\pi^2 . \end{aligned} \tag{7}$$

Thus, using definitions (3), (4), and (5),

$$\begin{aligned} Q_n^2 &= 2\mu_\pi^2 \sum_{i<j}^{n} [C_{2L}^2(i,j) + C_{2T}^2(i,j) + C_{2R}^2(i,j)] \qquad (8) \\ &= 2\mu_\pi^2 (C_{nL}^2 + C_{nT}^2 + C_{nR}^2), \qquad (9) \end{aligned}$$

where

$$C_{nX}^2 = \sum_{i<j}^{n} C_{2X}^2(i,j) \ \text{ with } \ X = L,\ T, \text{ or } R. \tag{10}$$

These generalized coalescence variables can be calculated for a system composed of any number of pions, given the momentum components of each particle of the system. The coalescence variables for all particle numbers have the same significance, denoting the amount by which the momentum mismatch along a particular axis increases the invariant mass of the system, in units of μ_π. Thus, as we will see, coalescence variables can be useful in comparing n-particle correlations over a range of values of n.

For the purposes of the comparisons presented in this paper we will use a simple linear parametrization of the momentum variables in terms of the kinematic parameters y_0, δy, m_0, δm, ϕ_0, and $\delta\phi$, the central value and difference of the rapidity, transverse mass, and azimuthal angle, respectively. For the two-particle case, we will take y_1=$y_0 + \delta y$, y_2=$y_0 - \delta y$, m_1=$m_0 + \delta m$, m_2=$m_0 - \delta m$, ϕ_1=$\phi_0 + \delta\phi$, and ϕ_2=$\phi_0 - \delta\phi$. Thus m_0=$(m_1 + m_2)/2$ and δm=$(m_1 - m_2)/2$, etc. For the three-particle case, we will take y_1=$y_0 + \delta y'$, y_2=y_0, y_3=$y_0 - \delta y'$, m_1=$m_0 + \delta m'$, m_2=m_0, m_3=$m_0 - \delta m'$, ϕ_1=ϕ_0+$\delta\phi'$, ϕ_2=ϕ_0, and ϕ_3=ϕ_0−$\delta\phi'$, where $\delta y'$=0.86δy, $\delta m'$=0.86δm, and $\delta\phi'$=0.86$\delta\phi$. For the four-particle case, we will take y_1=$y_0 + \delta y''$, y_2=$y_0 + \delta y''/3$, y_3=$y_0 - \delta y''/3$, y_4=$y_0 - \delta y''$, m_1=$m_0 + \delta m''$, m_2=$m_0 + \delta m''/3$, m_3=$m_0 - \delta m''/3$, m_4=$m_0 - \delta m''$, ϕ_1=$\phi_0 + \delta\phi''$, ϕ_2=$\phi_0 + \delta\phi''/3$, ϕ_3=$\phi_0 - \delta\phi''/3$, and ϕ_4=$\phi_0 - \delta\phi''$, where $\delta y''$=0.72δy, $\delta m''$=0.72δm, and $\delta\phi''$=0.72$\delta\phi$.

Reduction factors have been applied to the $n = 3$ and $n = 4$ difference parameters above to give the corresponding coalescence variables about the same dependence as

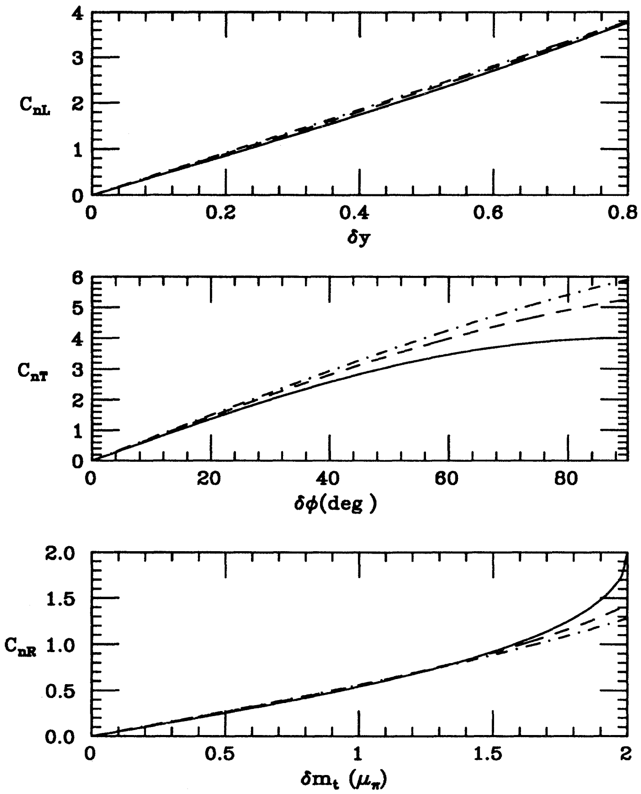

FIG. 1. The coalescence variables C_{nL}, C_{nT}, and C_{nR} for $n = 2$ (solid curves), $n = 3$ (dashed curves), and $n = 4$ (dot-dashed curves) are plotted against the kinematic difference parameters δy, $\delta\phi$, and δm for $y_0=\phi_0=0$, and $m_0=3\mu_\pi$.

the $n = 2$ coalescence variables. Figure 1 shows plots of the coalescence variables C_{nL}, C_{nT}, and C_{nR} plotted against the difference parameters δy, $\delta\phi$, and δm for $y_0=\phi_0=0$, $n = 2$, 3, and 4, and $m_0=3\mu_\pi$. We see that all three coalescence variables are single valued monotonically increasing functions of δy, $\delta\phi$, and δm and that C_{2X}, C_{3X}, and C_{4X} ($X = L$, T, and R) have similar behavior and normalizations.

IV. APPLICATION TO 2, 3, AND 4-PARTICLE CORRELATIONS

Up to now, investigations of the correlations of pions produced in ultrarelativistic heavy ion collisions have focused on two particle systems, with a few studies of three-particle correlations. However, we expect that with the higher multiplicities expected collisions at RHIC and LHC energies, correlations with larger numbers of pions (and kaons) will play a prominent role in the analysis. Here we will use the coalescence variables to compare the correlations of two to four pions.

We are particularly interested in the competition between the rising strength due to Bose-Einstein attraction exhibited by the general n-particle correlation R_n, which for neutral particles has a maximum value of $n!$ when the overall coalescence $Q^2 = 0$, and the strong suppression of the charged pion correlation $R_n^{[\pm]}$ due to Coulomb repulsion, which behaves approximately as $F(\eta_{\rm avg})^{n(n-1)/2}$ due to the mutual repulsion of charged pions that are closely correlated. Here F is the Gamow penetrability and $\bar{\eta}$ the average Sommerfeld parameter, both defined below, which characterize the Coulomb interactions of the system. This competition ultimately depends on formidable and unresolved theoretical issues, particularly the derivation of a reliable expression for multiparticle Coulomb effects. In the present work the Bose-Einstein/Coulomb competition will be investigated through the use of an analytic model for the correlations and a simple but plausible Coulomb correction procedure.

Kolehmainen and Gyulassy[7,8] have presented a boost-invariant analytic model for predicting the one- and two-particle correlations of pions from an ultrarelativistic heavy ion collision. The model is able to obtain an analytic expression for the correlation distribution by employing a simple pseudothermal description of the pion momentum distribution function. It uses the inside-outside cascade model[9] to describe the pion emission process and assumes that the emissions are completely incoherent. It characterizes the pion-emitting volume with three parameters, the transverse radius r_T of the emitting volume, and the time constant τ_0 of the emission process, and the pseudo-temperature T of the source. The pseudo-temperature of the Kolehmainen and Gyulassy model is related to the physical temperature U of a thermal model by the relation $T = 1.42U - 12.7$ MeV. In the calculations below, we will usually take $U{=}\mu_\pi$, which is equivalent to T=186.1 MeV.

The one-particle inclusive pion yield given by this model is

$$\frac{d^3N_\pi}{d^3P_1} = AK_0(m_1/T) \ , \tag{11}$$

where A is a normalization factor and $K_0(z)$ is a modified irregular cylindrical Bessel function of order zero with possibly complex argument z. We will also consider the n-particle correlation distributions, here defined as the ratio of the n-particle inclusive pion yield to the product of n one-particle inclusive pion yields, i.e.,

$$R_n(\mathbf{P}_1,\ldots,\mathbf{P}_n) = \frac{d^{3n}N_\pi}{d^3P_1\cdots d^3P_n} \Big/ \prod_{i=1}^{n} \frac{d^3N_\pi}{d^3P_i} \ . \tag{12}$$

The two-particle correlation distribution for neutral particles (superscript [0]), in the Kolehmainen-Gyulassy model, is given by

$$R_2^{[0]}(\mathbf{P}_1,\mathbf{P}_2) = 1 + B_{12}^2 \ , \tag{13}$$

where

$$B_{12} = \frac{|G(\mathbf{P}_1,\mathbf{P}_2)|}{\sqrt{G(\mathbf{P}_1,\mathbf{P}_1)G(\mathbf{P}_2,\mathbf{P}_2)}} \ , \tag{14}$$

and

$$G(\mathbf{P}_1,\mathbf{P}_2) = AK_0(\sqrt{u_{12}})\exp\left[-\Delta p_{12}^2\left(\frac{r_T}{2\hbar}\right)^2\right] , \qquad (15)$$

with

$$\Delta p_{12}^2 = p_1^2 + p_2^2 - 2p_1p_2\cos(\phi_1-\phi_2) , \qquad (16)$$

and

$$u_{12} = \left[\frac{m_1+m_2}{2T} + \frac{i\tau_0}{\hbar}(m_1-m_2)\right]^2 + 4\left[\left(\frac{1}{2T}\right)^2 + \left(\frac{\tau_0}{\hbar}\right)^2\right] m_1m_2\sinh^2\left|\frac{y_1-y_2}{2}\right| . \qquad (17)$$

In terms of the coalescence variables we can write (16) and (17) as

$$\Delta p_{12}^2 = (m_1+m_2)^2 + 2\mu_\pi^2(C_{2R}^2 + C_{2T}^2), \qquad (18)$$

and

$$u_{12} = \left[\frac{m_1+m_2}{2T} + \frac{i\tau_0}{\hbar}(m_1-m_2)\right]^2 +2\left[\left(\frac{1}{2T}\right)^2 + \left(\frac{\tau_0}{\hbar}\right)^2\right]\mu_\pi^2 C_{2L}^2 . \qquad (19)$$

Note that u_{12} is a complex quantity that becomes purely real when $(m_1 - m_2)=0$. The quantity $(m_1 - m_2)$ in Eq. (19) is closely related to the radial coalescence C_{2R}, but no simple function of the latter can be used in the equation.

In the present work we have generalized the Kolehmainen-Gyulassy model summarized in Eqs. (11)–(19) in several ways. First, following the work of Biyajima *et al.*,[11] we have used quantum optics (QO) interference diagrams to derive the two-, three-, and four-particle correlation distributions for neutral particles in terms of the Kolehmainen-Gyulassy model with the addition of coherence. The Biyajima formalism includes the possibility of a coherent contribution to particle emission, but implicitly assumes that there is *only one* source of coherent emission. The two-particle correlation function, calculated in this way, is a generalization of Eq. (13) and has the following form:

$$R_2^{[0]}(\mathbf{P}_1,\mathbf{P}_2) = 1 + \epsilon^2 B_{12}^2 + 2\epsilon(1-\epsilon)B_{12} . \qquad (20)$$

Here the parameter ϵ specifies the fraction of the net emission of the source that is incoherent and is defined as $\epsilon = m_{\text{incoh}}/(m_{\text{incoh}} + m_{\text{coh}})$, where m_x is the pion multiplicity of type x. When coherent emission of pions is significant, $\epsilon < 1$ and this has the effect of reducing the peaking near $Q^2=0$ of the correlation distributions. In the calculated examples presented later in this paper we will assume that the pion source is completely incoherent, i.e., $\epsilon=1$.

The three neutral particle correlation derived from QO diagrams is

$$\begin{aligned} R_3^{[0]}(\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3) = 1 &+\epsilon^2(B_{12}^2 + B_{23}^2 + B_{31}^2) + 2\epsilon^3(B_{12}B_{23}B_{31}) \\ &+2\epsilon(1-\epsilon)(B_{12}+B_{23}+B_{31}) + 2\epsilon^2(1-\epsilon)(B_{12}B_{23} + B_{13}B_{32} + B_{21}B_{13}) , \end{aligned} \qquad (21)$$

and the four neutral particle correlation is

$$\begin{aligned} R_4^{[0]}(\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3,\mathbf{P}_4) = {} & 1 + \epsilon^2(B_{12}^2 + B_{13}^2 + B_{14}^2 + B_{23}^2 + B_{24}^2 + B_{34}^2) \\ & + 2\epsilon^3(B_{12}B_{23}B_{31} + B_{12}B_{24}B_{41} + B_{13}B_{34}B_{41} + B_{23}B_{34}B_{42}) \\ & + 2\epsilon^4(B_{12}B_{23}B_{34}B_{41} + B_{12}B_{24}B_{43}B_{31} + B_{13}B_{32}B_{24}B_{41}) \\ & +\epsilon^4(B_{12}^2B_{34}^2 + B_{13}^2B_{24}^2 + B_{14}^2B_{23}^2) + 2\epsilon(1-\epsilon)(B_{12}+B_{13}+B_{14}+B_{23}+B_{24}+B_{34}) \\ & + 3\epsilon^2(1-\epsilon)(B_{12}B_{23} + B_{12}B_{24} + B_{13}B_{34} + B_{23}B_{34} + B_{12}B_{31} + B_{12}B_{41} + B_{13}B_{41} + B_{23}B_{42}) \\ & + 4\epsilon^2(1-\epsilon)^2(B_{12}B_{34} + B_{13}B_{24} + B_{14}B_{23}) \\ & + 2\epsilon^3(1-\epsilon)(B_{12}B_{23}B_{34} + B_{12}B_{24}B_{43} + B_{13}B_{32}B_{24} + B_{12}B_{23}B_{41} + B_{12}B_{24}B_{31} + B_{13}B_{32}B_{41} \\ & \qquad + B_{12}B_{34}B_{41} + B_{12}B_{43}B_{31} + B_{13}B_{24}B_{41} \\ & \qquad + B_{23}B_{34}B_{41} + B_{24}B_{43}B_{31} + B_{32}B_{24}B_{41}) \\ & + 2\epsilon^3(1-\epsilon)(B_{12}B_{34}^2 + B_{13}B_{24}^2 + B_{14}B_{23}^2 + B_{12}^2B_{34} + B_{13}^2B_{24} + B_{14}^2B_{23}). \end{aligned} \qquad (22)$$

Note that for completely incoherent systems, the $(1-\epsilon)$ terms in the above relations will vanish, resulting in considerable simplification of the equations.

As a second generalization of the Kolehmainen-Gyulassy model, we have included an approximate correction for the mutual Coulomb repulsion of the emitted identical particles. This allows us to obtain $R_n^{[\pm]}(\mathbf{P}_1,\ldots,\mathbf{P}_n)$, the n-particle correlation distributions for *charged* particles (superscript [±]). Following Gyulassy*et al.*,[10] we have used for two charged-particle correlations a Coulomb correction of the following form:

$$R_2^{[\pm]}(\mathbf{P}_1,\mathbf{P}_2) = F(\eta_{12})R_2^{[0]}(\mathbf{P}_1,\mathbf{P}_2) , \qquad (23)$$

where $F(\eta_{12})$ is the Gamow penetrability function $2\pi\eta_{12}/[\exp(2\pi\eta_{12})-1]$ and η_{12} is the Sommerfeld parameter α/β_{12}. Here α is the fine structure constant and β_{12} is the velocity of particle 1 relative to particle 2 divided by the velocity of light. For the case of pions with kinetic energies comparable to μ_π, the quantity $2\pi\eta$ in the penetrability function is typically much less than 1, which implies that an expansion of the Coulomb correction in powers of $2\pi\eta$ should converge rapidly. As a correction for final-state-interaction effects, the use of the Gamow penetrability is a valid approximation only when non-Coulomb final-state interactions can be neglected and when the Bohr radius of the system of two charged particles is large compared to the dimensions of the emission source. For pions, the two-particle-distance scale implied by the Bohr radius is about 193 fm, a large distance compared to expected source sizes of 4 to 40 fm.

Liu *et al.*,[4] have suggested an *ad hoc* product-form Coulomb correction for a system of three charged particles of the form $[F(\eta_{12})F(\eta_{23})F(\eta_{31})]$. Here $F(\eta_{ij})$ is the Gamow penetrability, and η_{ij} the Sommerfeld parameter describing the Coulomb interaction of particles i and j. We can provide some justification for a correction of this form by considering the following *gedanken experiment*: Let us "assemble" an n-pion system by bringing the pions one at a time from infinite relative momentum down to the momentum state appropriate to the η value of the new pion with respect to the already assembled pions. In this case the net penetrability of the assembled system, and therefore the suppression of the correlation, will be of the form $[F(\bar{\eta})F(2\bar{\eta})F(3\bar{\eta})\cdots F((n-1)\bar{\eta})]$, where $\bar{\eta}$ is some average Sommerfeld parameter of the various pion pairs. For two charged pions, this is just the Coulomb correction given in Eq. (23). For three charged pions, this leads to a Coulomb correction of the form $[F(\bar{\eta})F(2\bar{\eta})]$. To order η^2, this can be approximated by $[F(\bar{\eta})]^3$, essentially the correction suggested by Liu *et al.*[4] However, a better approximation that is valid to order η^7 can be obtained by introducing η', the $n = 3$ effective Sommerfeld parameter, which is defined by the relation $F(\eta)F(2\eta)/[F(\eta')]^3 = 1 + \mathcal{O}(\eta^7)$. We use the series expansion

$$x' = x + \tfrac{1}{9}x^2 - \tfrac{1}{27}x^3 - \tfrac{19}{1215}x^4 + \tfrac{11}{1215}x^5 + \tfrac{967}{229\,635}x^6,$$

where $x = \pi\eta$ and $x' = \pi\eta'$. The Coulomb correction can then be symmetrized in the particle indices to give a correction valid to order η^7 that has the form $[F(\eta'_{12})F(\eta'_{23})F(\eta'_{31})]$. The three-particle correlation function for charged particles is then

$$R_3^{[\pm]}(\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3) = [F(\eta'_{12})F(\eta'_{23})F(\eta'_{31})]\,R_3^{[0]}(\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3)\,, \quad (24)$$

where η'_{ij} is the $n = 3$ effective Sommerfeld parameter defined above.

This conceptual approach to Coulomb corrections can be extended to a system of four charged particles, where it has the form $F(\bar{\eta})F(2\bar{\eta})F(3\bar{\eta})$. To order η^2, this can be approximated by $[F(\bar{\eta})]^6$, but a better approximation valid to order η^7 can be obtained by introducing η'', the $n = 4$ effective Sommerfeld parameter, which is defined by the relation $F(\eta)F(2\eta)F(3\eta)/[F(\eta'')]^6 = 1 + \mathcal{O}(\eta^7)$. We use the series expansion

$$x'' = x + \tfrac{2}{9}x^2 - \tfrac{2}{27}x^3 - \tfrac{167}{2430}x^4 + \tfrac{1}{30}x^5 + \tfrac{18\,091}{459\,270}x^6,$$

where $x = \pi\eta$ and $x'' = \pi\eta''$. The Coulomb correction can then be symmetrized in the particle indices to give a correction valid to order η^7 of the following form:

$$[F(\eta''_{12})F(\eta''_{13})F(\eta''_{14})F(\eta''_{23})F(\eta''_{24})F(\eta''_{34})],$$

where η''_{ij} is the $n = 4$ effective Sommerfeld parameter defined above. The four-particle correlation for charged particles is then

$$R_4^{[\pm]}(\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3,\mathbf{P}_4) = [F(\eta''_{12})F(\eta''_{13})F(\eta''_{14})F(\eta''_{23})F(\eta''_{24})F(\eta''_{34})] \times R_4^{[0]}(\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3,\mathbf{P}_4)\,. \quad (25)$$

The Coulomb corrections of Eqs. (24) and (25) are based on a somewhat schematic conceptual model, the use of which is justified only because a more accurate multiparticle Coulomb correction is not available. We note, however, that both of these Coulomb corrections exhibit an important and required property: When one of n correlated particles x is made electrically neutral or given a

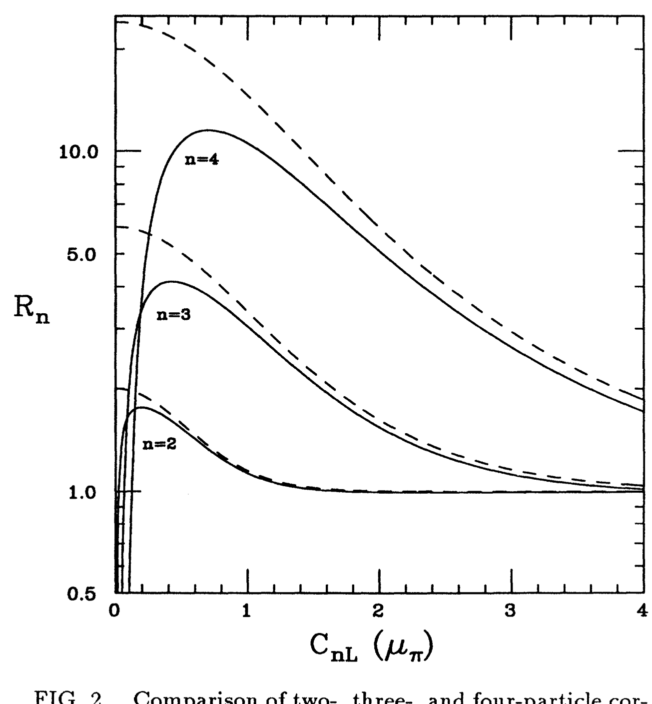

FIG. 2. Comparison of two-, three-, and four-particle correlation distributions for neutral particles $R_n^{[0]}$ (dashed curves) and for charged particles $R_n^{[\pm]}$ (solid curves) as functions of the longitudinal coalescence C_{nL}. The parameters used are given in the text.

large relative velocity [so that all of its penetrability factors $F(\eta_{ix}) \rightarrow 1$], the Coulomb correction reduces to that appropriate to the next lowest order correlation, i.e., the correction for $n-1$ particles.

Figure 2 shows a comparison of neutral and charged two-, three, and four-particle correlation distributions $R_n^{[\pm,0]}$ as functions of the longitudinal coalescence. The fixed parameters for this plot are $y_0=\phi_0=\delta m=\delta\phi=0$, $\epsilon=1$, $m_0=3\mu_\pi$, r_T=1.0 fm, τ_0=2.0 fm/c, and T = 185.5 MeV. The longitudinal coalescence was varied over $0 \leq C_{nL} \leq 4$. The dashed lines show the neutral-particle correlation and the solid lines show the charged-particle correlations. The effect of the Coulomb force in suppressing the correlation at small relative momenta is apparent in these curves, as is the greater enhancement over uncorrelated background (S_n=1) of the n-particle correlation distribution.

Figure 3 shows a comparison of neutral and charged two-, three-, and four-particle correlation distributions $R_n^{[\pm,0]}$ as functions of the transverse coalescence. The fixed parameters for these calculated functions are $y_0=\delta y=\phi_0=\delta m=0$, $\epsilon=1$, $m_0=3\mu_\pi$, r_T=1.0 fm, τ_0=2.0 fm/c, and T = 185.5 MeV. The transverse coalescence was varied over $0 \leq C_{nT} \leq 4$.

Figure 4 shows a similar comparison of neutral and charged two-, three-, and four-particle correlation distributions $R_n^{[\pm,0]}$ as functions of the radial coalescence. The fixed parameters for these calculated functions are $y_0=\delta y=\phi_0=\delta\phi=0$, $\epsilon=1$, $m_0=3\mu_\pi$, r_T=1.0 fm, τ_0=2.0 fm/c, and T = 185.5 MeV. The radial coalescence was varied over $0 \leq C_{nR} \leq 1.4$.

V. DEPENDENCE ON TRANSVERSE RADIUS R_T

Equations (15) and (18) imply that the function $G(\mathbf{P}_1,\mathbf{P}_2)$ has a term that is a Gaussian exponential with an exponent of the form $-\frac{1}{2}[(\mu_\pi/\hbar)r_T C_{2T}]^2$ and a similar term involving C_{2R}. This means that the transverse has a complementary relationship with both C_{nT} and C_{nR}; i.e., the widths of the correlation peaks in these coalescence variables depend on $1/r_T$. This dependence is illustrated in Figs. 5 and 6, in which the transverse radius is varied to show its effect in the transverse and radial correlation peaks. The longitudinal correlations are not shown because the longitudinal correlation peak widths are independent of r_T.

Figure 5 shows the charged 2-, 3-, and 4-particle correlation distributions $R_n^{[\pm]}$ as functions of the transverse coalescence C_{nT} over a range of values of r_T, the transverse radius of the source. The fixed parameters for these calculated functions are $y_0=\phi_0=\delta m=\delta y=0$, $\epsilon=1$, $m_0=3\mu_\pi$, τ_0=4 fm/c, and T = 185.5 MeV. The transverse coalescence was varied over $0 \leq C_{nT} \leq 1.8$. The separate curves correspond to transverse radius value of r_T = 1 (highest curve), 2, 4, 8, 16, and 32 fm (lowest curve).

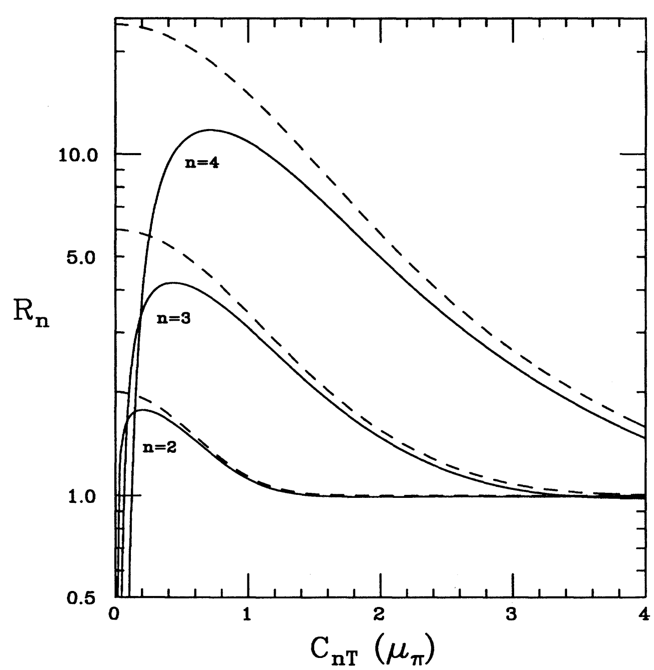

FIG. 3. Comparison of two-, three-, and four-particle correlation distributions for neutral particles $R_n^{[0]}$ (dashed curves) and for charged particles $R_n^{[\pm]}$ (solid curves) as functions of the transverse coalescence C_{nT}. The parameters used are given in the text.

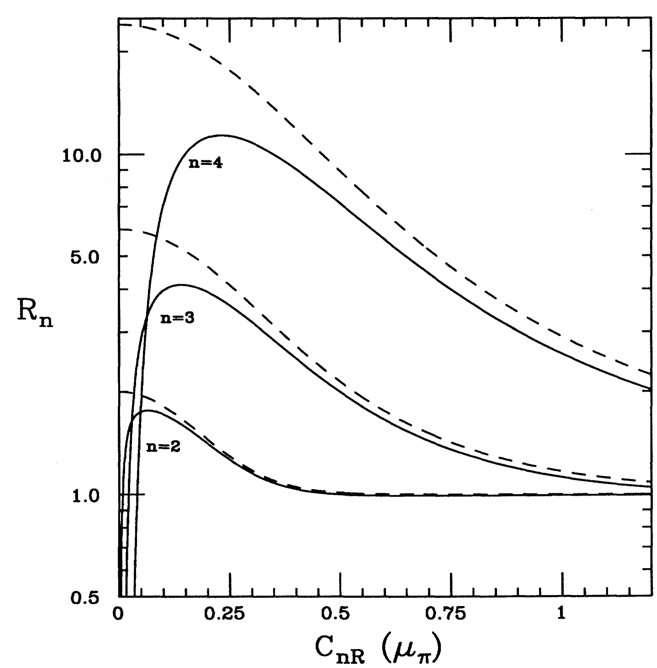

FIG. 4. Comparison of two-, three-, and four-particle correlation distributions for neutral particles $R_n^{[0]}$ (dashed curves) and for charged particles $R_n^{[\pm]}$ (solid curves) as functions of the radial coalescence C_{nR}. The parameters used are given in the text.

Figure 6 shows the charged 2-, 3-, and 4-particle correlation distributions $R_n^{[\pm]}$ as functions of the radial coalescence C_{nR} over a range of values of r_T, the transverse radius of the source. The fixed parameters for these calculated functions are $y_0=\phi_0=\delta m=\delta y=0$, $\epsilon=1$, $m_0=3\mu_\pi$, $\tau_0=4$ fm/c, and $T = 185.5$ MeV. The radial coalescence was varied over $0 \leq C_{nR} \leq 0.4$. The separate curves correspond to transverse radius value of $r_T = 1$ (highest curve), 2, 4, 8, 16, and 32 fm (lowest curve).

We note in Figs. 5 and 6 that while the peaking of the correlation distributions is very strong when the source radius is small, it becomes progressively weaker for larger radii. This is because the peak of the correlation distribution is forced into the small coalescence region which is dominated by strong Coulomb repulsion. The extraction of transverse source radii with charged pion Hanbury-Brown–Twiss correlations therefore will not be possible when the transverse radius of the emitting source approaches a value of $r_T \approx 32$ fm.

VI. DEPENDENCE ON TIME CONSTANT τ_0

Because the modified Bessel function $K_0(z)$ has the asymptotic behavior of an exponential in z^2, Eqs. (15) and (19) imply that the function $G(\mathbf{P}_1, \mathbf{P}_2)$ contains a term, which is a Gaussian exponential with an exponent of the form $-2[(\mu_\pi/\hbar)\tau_0 C_{2L}]^2$, and a similar term involving $(m_1 - m_2)$, which is related to C_{2R}. This means that the emission time constant τ_0 has a complementary relationship with C_{nL} and possibly also with C_{nR}. This dependence is illustrated in Figs. 7 and 8, in which the transverse radius is varied to show its effect in the longitudinal and radial correlation peaks. The transverse correlations are not shown because its correlation peak width is independent of τ_0.

Figure 7 shows the charged 2-, 3-, and 4-particle correlation distributions $R_n^{[\pm]}$ as functions of the longitudinal coalescence C_{nL} over a range of values of τ_0, the emission time constant of the source. The fixed parameters for these calculated functions are $y_0=\phi_0=\delta m=\delta\phi=0$, $\epsilon=1$, $m_0=3\mu_\pi$, $r_T= 10$ fm, and $T = 185.5$ MeV. The longitudinal coalescence was varied over $0 \leq C_{nL} \leq 1.8$. The separate curves correspond to source duration values of $\tau_0 = 1$ (highest curve), 2, 4, 8, 16, and 32 fm/c (lowest curve).

Figure 8 shows the charged 2-, 3-, and 4-particle correlation distributions $R_n^{[\pm]}$ as functions of the radial coalescence C_{nR} over a range of values of τ_0, the emission

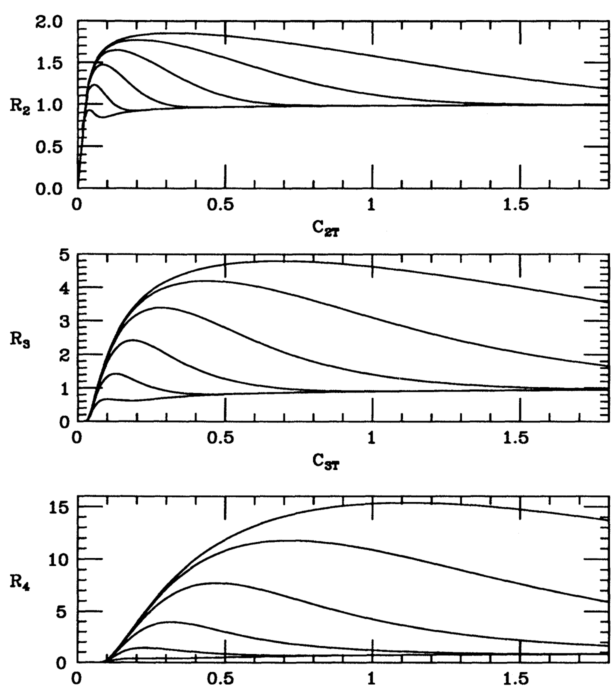

FIG. 5. The 2, 3, and 4 charged-particle correlation distributions $R_n^{[\pm]}$ as functions of the transverse coalescence C_{nT} over a range of values of r_T, the transverse radius of the source. The parameters used are given in the text. The separate curves correspond to transverse radii of $r_T = 1$ (highest curve), 2, 4, 8, 16, and 32 (lowest curve) fm.

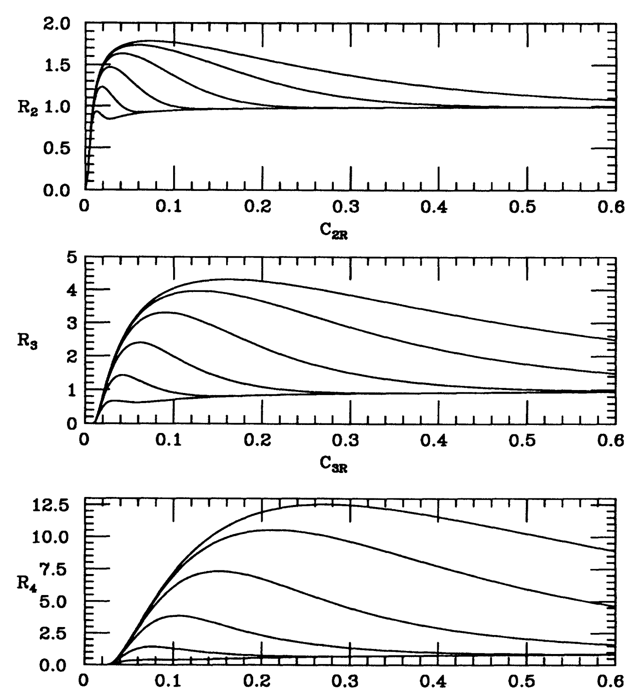

FIG. 6. The 2, 3, and 4 charged-particle correlation distributions $R_n^{[\pm]}$ as functions of the radial coalescence C_{nR} over a range of values of r_T, the transverse radius of the source. The parameters used are given in the text. The separate curves correspond to transverse radii of $r_T = 1$ (highest curve), 2, 4, 8, 16, and 32 (lowest curve) fm.

time constant of the source. The fixed parameters for these calculated functions are $y_0=\phi_0=\delta m=\delta\phi=0$, $\epsilon=1$, $m_0=3\mu_\pi$, $r_T=$ 10 fm, and T = 185.5 MeV. The radial coalescence was varied over $0 \leq C_{nR} \leq 0.4$. The separate curves correspond to source duration values of $\tau_0=1$ (highest curve), 2, 4, 8, 16, and 32 fm/c (lowest curve).

Figures 7 and 8 show a problem in the extraction of the source emission time constant which is similar to that encountered in extracting the source radius but less severe. When the transverse radius is about $r_T \approx 10$ fm, the correlation peak disappears in the longitudinal correlation (Fig. 7) but persists in the radial correlation (Fig. 8). This rather unexpected result could permit extraction of τ_0 in systems with moderate transverse radii but very large emission time constants, if such sources exist.

VII. DEPENDENCE ON PSEUDO-TEMPERATURE T

Equations (15) and (19) also imply that T, the pseudo-temperature of the source will have some effect on both the longitudinal and radial correlations. This dependence is illustrated in Figs. 9 and 10, in which the pseudo-temperature is varied by a factor of 4 to show its effect on the longitudinal and radial correlation peaks. The transverse correlations are not shown because, in the Kolehmainen-Gyulassy model, the correlation peak width is independent of temperature.

Figure 9 shows the charged 2-, 3-, and 4-particle correlation distributions $R_n^{[\pm]}$ as functions of the longitudinal coalescence C_{nL} over a range of values of T, the pseudo-temperature of the source. The fixed parameters for these calculated functions are $y_0=\phi_0=\delta m=\delta\phi=0$, $\epsilon=1$, $m_0=3\mu_\pi$, $r_T=$ 10 fm, and $\tau_0=4$ fm/c. The longitudinal coalescence was varied over $0 \leq C_{nL} \leq 1.8$. The separate curves correspond to pseudo-temperature values of T=186.1 (highest curve), 268.4, 384.9, 549.5, and 782.6 (lowest curve) MeV, which correspond to physical temperatures of $U=\mu_\pi 2^{n/2}$ with n=0 to 4.

Figure 10 shows the charged 2-, 3-, and 4-particle correlation distributions $R_n^{[\pm]}$ as functions of the radial coalescence C_{nR} over a range of values of T, the pseudo-temperature of the source. The fixed parameters for these calculated functions are $y_0=\phi_0=\delta m=\delta\phi=0$, $\epsilon=1$, $m_0=3\mu_\pi$, r_T=10 fm, and τ_0=4 fm/c. The radial coalescence was varied over $0 \leq C_{nR} \leq 1.8$. The separate curves correspond to pseudo-temperature values of T=186.1 (highest curve), 268.4, 384.9, 549.5, and 782.6 MeV (lowest curve), which correspond to physical tem-

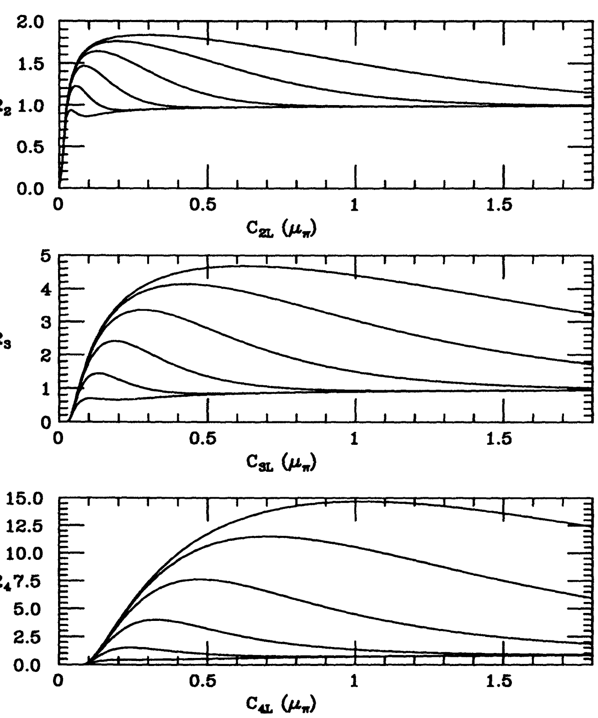

FIG. 7. The 2, 3, and 4 charged-particle correlation distributions $R_n^{[\pm]}$ as functions of the longitudinal coalescence C_{nL} over a range of values of τ_0, the emission time constant of the source. The parameters used are given in the text. The separate curves correspond to source emission time constants of $\tau_0=1$ (highest curve), 2, 4, 8, 16, and 32 (lowest curve) fm/c.

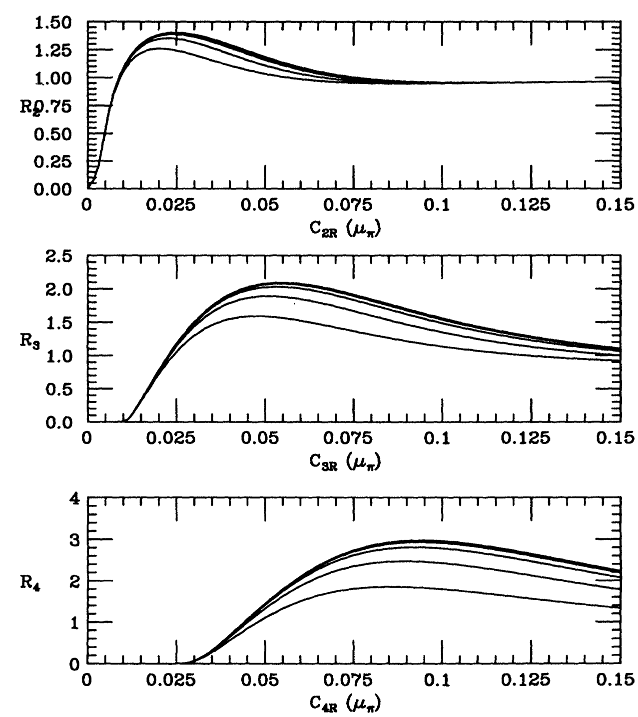

FIG. 8. The 2, 3, and 4 charged-particle correlation distributions $R_n^{[\pm]}$ as functions of the radial coalescence C_{nR} over a range of values of τ_0, the emission time constant of the source. The parameters used are given in the text. The separate curves correspond to source emission time constants of $\tau_0=1$ (highest curve), 2, 4, 8, 16, and 32 (lowest curve) fm/c.

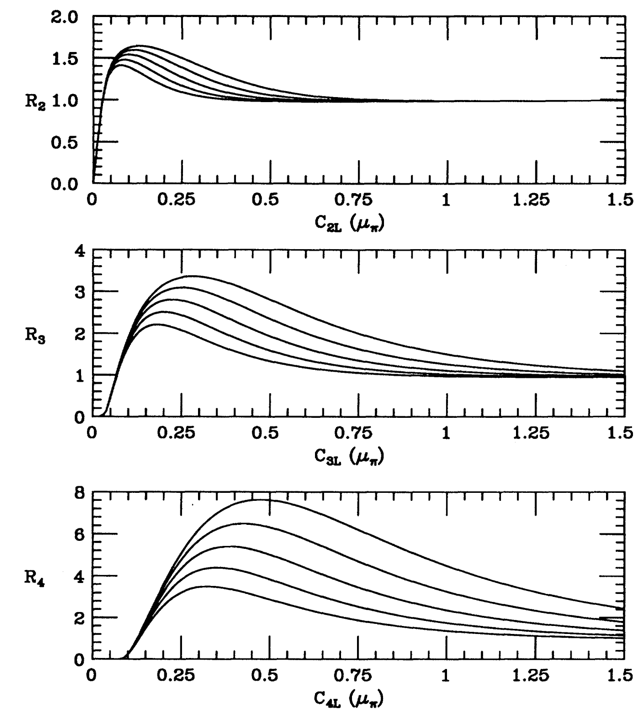

FIG. 9. The charged 2-, 3-, and 4-particle correlation distributions $R_n^{[\pm]}$ as functions of the longitudinal coalescence C_{nL} over a range of values of T, the pseudo-temperature of the source. The parameters used are given in the text. The separate curves correspond to pseudo-temperature values of T=186.1 (highest curve), 268.4, 384.9, 549.5, and 782.6 (lowest curve) MeV, which correspond to physical temperatures of 4.

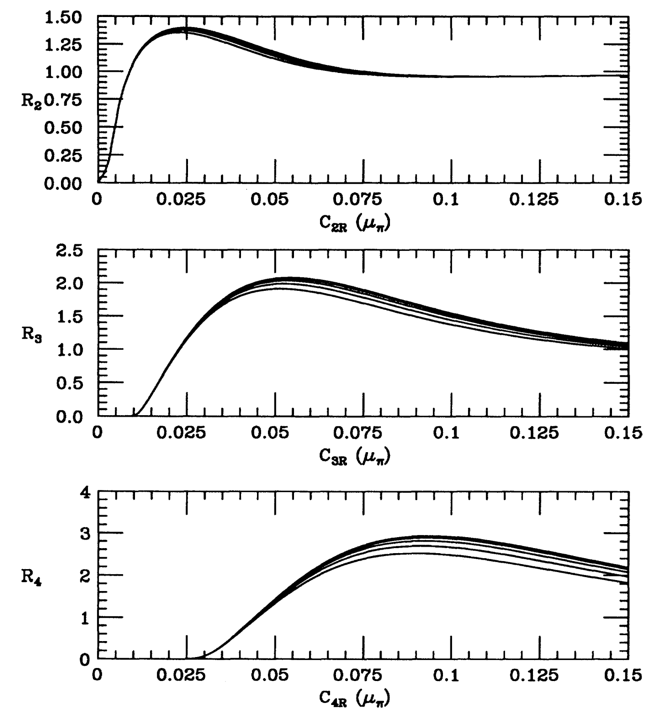

FIG. 10. The charged 2-, 3-, and 4-particle correlation distributions $R_n^{[\pm]}$ as functions of the radial coalescence C_{nR} over a range of values of T, the pseudo-temperature of the source. The parameters used are given in the text. The separate curves correspond to pseudo-temperature values of T=186.1 (highest curve), 268.4, 384.9, 549.5, and 782.6 (lowest curve) MeV, which correspond to physical temperatures of U=$\mu_\pi 2^{n/2}$ with n = 0 to 4.

peratures of U=$\mu_\pi 2^{n/2}$ with n=0 to 4.

It is interesting to note in this comparison that elevating the temperature, which shifts the pion momentum spectrum to higher energies, also has the effect of reducing both the longitudinal and the radial correlation peaks.

VIII. CONCLUSION

We have introduced a new set of Lorentz-invariant kinematic variables, the longitudinal, transverse, and radial coalescence, for describing the relative state of a group of pi mesons that are analyzed with an n-particle correlation distribution R_n. The value of these variables has been illustrated in the examples above, it can be summarized as follows:

The coalescence variables are dimensionless, Lorentz-invariant, monotonically increasing functions of the more familiar relative rapidity y, transverse angle ϕ, and transverse mass m_T of the correlated pions.

The coalescence variables allow direct comparisons among the correlation distributions for any number of pions.

The coalescence variables for a multiparticle system are formed by adding the corresponding two-particle coalescence variables in quadrature for all particle pairs present in the system.

The coalescence variables provide a Lorentz-invariant separation of the timelike and spacelike characteristics of the source of particles.

We have extended the analytic Kolehmainen-Gyulassy model by including the effects of source coherence and of Coulomb repulsion between the correlated pions. For the correlation distributions of order n using the Kolehmainen-Gyulassy model, the longitudinal coalescence C_{nL} has a complementary relation with the source duration τ_0, the transverse coalescence C_{nT} has a complementary relation with the source radius r_T, and the radial coalescence C_{nR} has a complementary relation with a combination of τ_0 and r_T. The results of this investigation can be summarized as follows:

For source sizes greater than about $r_T \approx 32$ fm, the two-, three-, and four-particle correlation distributions peak in a region of coalescence where they are completely suppressed by Coulomb repulsion, making correlations of charged pions in this size domain insensitive to source size.

It is apparently possible to extract emission time constants with values $\tau_0 \approx 32$ fm by analyzing the correlation dependence on radial coalescence, provided the transverse radius of the source is of moderate size.

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