# THE REDUCED ROTATION MATRIX: PLOTS AND ZEROS*

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The reduced matrix elements of finite rotations in the angular momentum representation $\{d^j_{m'm}(\beta)\}$ are presented in polar plots and root tables for several low spins having both integral and half-integral values. The plots present, for $j \le 5/2$, the magnitude of the matrix element as the length of the radius vector at the polar angle which corresponds to the Euler angle of rotation. The tables present those values of $\beta$, in the region $0 \le j \le 8$, which satisfy the condition $d^j_{m'm}(\beta) = 0$.

## CONTENTS

## INTRODUCTION

Although an extensive tabulation of the reduced rotation matrix $d^j_{mm'}(\beta)$ is available in the literature[1] there are no convenient plots available between 0 and $2\pi$ for low values of both integer and half-integer spin. Rough values of $d^j_{mm'}(\beta)$ may be obtained from polar graphs, but their major use is in providing a clearer visualization of the symmetries (for a particular spin) than is obtained from ordinary graphical form. Further, there is only one short tabulation (for integer spins *only*) of zeros of the reduced rotation matrix available in the literature[2] despite the important role these zeros play in the theory of angular correlations.[2,3]

The reduced rotation matrix is evaluated using a Wigner closed-form[4] expression:

$$d^j_{mm'}(\beta) \qquad (1)$$

$$= \left[ \frac{(j+m')! \,(j-m')!}{(j-m)! \,(j+m)!} \right]^{1/2} \sum_\sigma \binom{j+m}{j-m'-\sigma}\binom{j-m}{\sigma}$$

$$\times (-1)^{j-m-\sigma} \left(\cos\frac{\beta}{2}\right)^{2\sigma+m'+m}\left(\sin\frac{\beta}{2}\right)^{2j-2\sigma-m'-m}$$

where $\binom{N}{K} \equiv \dfrac{N!}{(N-K)!K!}$ is the binomial coefficient, $j$ is the angular momentum of the system, $m'$ the magnetic substate of $j$ before rotation, $m$ the magnetic substate after rotation, and $\beta$ the Euler angle of rotation.

The sum is over all values of $\sigma$ for which the factorials all have non-negative arguments.

The phase convention is that of Brink and Satchler.[5] Multiplication by $(-)^{m-m'}$ or interchange of subscripts $m$ and $m'$ will make the phase consistent with that of Behkami.[1]

The following symmetry relations exist:

$$d^j_{mm'}(\beta) \qquad (2)$$

$$= (-1)^{m-m'} d^j_{-m,-m'}(\beta) = (-1)^{j-m} d^j_{-mm'}(\pi+\beta)$$
$$= (-1)^{j-m'} d^j_{m,-m'}(\pi+\beta) = (-1)^{m-m'} d^j_{m'm}(\beta)$$
$$= d^j_{mm'}(-\beta).$$

## Comments and Results

A computer program[6] was used to perform the numerical evaluation of the elements of the reduced rotation matrix $d^j_{mm'}(\beta)$. In the present work, we have employed this program to generate plots of the reduced rotation matrix for $0 \leq j \leq 5/2$ and $|m,m'| \leq j$ with $0 \leq \beta \leq 2\pi$. The plots were done on the line printer of the University of Washington SDS 930 Computer System, employing special high-density plotting symbols

and the subroutine GRAPHIC which will be described elsewhere.[7]

The figure shows the polar plots of the matrix elements thereby generated. The representation employs a normal Cartesian coordinate system, with the $x$-axis to the right and $\beta$ increasing in the counter-clockwise direction. It was decided to make the radius vector equal in magnitude to the absolute value of the matrix elements and to indicate the sign of the matrix element by adding plus and minus signs in the different angular regions.

The program, together with ROOT, a program for finding the real roots of an equation numerically,[8] has been used to calculate the zeros of the reduced rotation matrix elements for values of $j$ up to $j = 8$. The table gives these zeros in *degrees*. It is interesting to note that there are certain regularities in these zeros which suggest symmetries in the rotation matrix which have not been described previously. However, a detailed discussion of these regularities is beyond the scope of the present paper.

*References*

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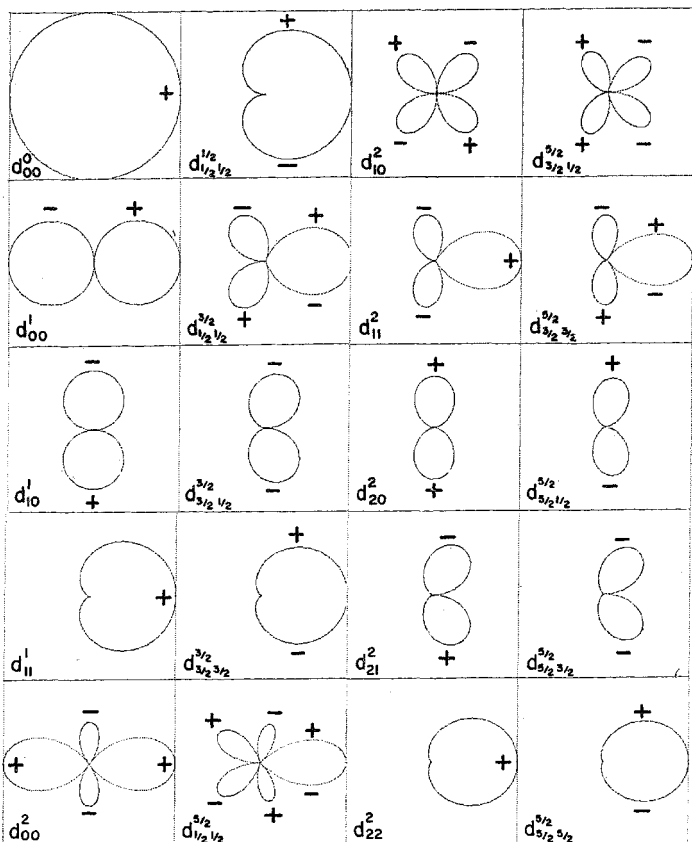
**Polar Plots of the Reduced Rotation Matrix**

The polar angle is the Euler angle of rotation of $d^j_{mm'}(\beta)$ and the length of the radius vector is the magnitude of this matrix element

The signs indicate the sign of the matrix element for $0 \leq \beta \leq 2\pi$. For integral $j$ the matrix element repeats every $2\pi$. For half-integral $j$ it repeats every $4\pi$. Signs for $2\pi \leq \beta \leq 4\pi$ can be found from the symmetry relations, Eq. (2)

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## TABLE. Zeros of the Reduced Rotation Matrix
### Explanation of Table

The table presents, in the columns headed ANGLES OF ZEROS, values of $\beta$ for which $d^j_{mm'}(\beta) = 0$. See Eq. (1) for $d^j_{mm'}(\beta)$ in the INTRODUCTION.

Only matrix elements with $m, m'$ positive and $m' \geq m$ were included because the missing matrix elements may be obtained from the symmetry relations of Eq. (2).

| J | j | Angular momentum of the system |
|---|---|---|
| MI | m' | Magnetic substate of $j$, before rotation |
| MF | m | Magnetic substate of $j$, after rotation |
| $\beta$ | | Euler angle of rotation |

One should be reminded here that $d^j_{mm'}(\beta)$ for half-integral $j$ repeats every $4\pi$ and for integral $j$ repeats every $2\pi$. However, in both cases a zero for $\beta$ implies a zero for $2\pi\text{-}\beta$. Therefore the tables are given only for $0 \leq \beta \leq \pi$.

TABLE. Zeros of the Reduced Rotation Matrix

| $J$ | $MF$ | $MI$ | Angles of Zeros | | | | |
|-----|------|------|-----|-----|-----|-----|-----|
| 0 | 0 | 0 | | | | | |
| 1/2 | 1/2 | 1/2 | 180.000 | | | | |
| 1 | 0 | 0 | 90.000 | | | | |
| 1 | 1 | 0 | .000 | 180.000 | | | |
| 1 | 1 | 1 | 180.000 | | | | |
| 3/2 | 1/2 | 1/2 | 70.529 | 180.000 | | | |
| 3/2 | 3/2 | 1/2 | .000 | 180.000 | | | |
| 3/2 | 3/2 | 3/2 | 180.000 | | | | |
| 2 | 0 | 0 | 54.736 | 125.264 | | | |
| 2 | 1 | 0 | .000 | 90.000 | 180.000 | | |
| 2 | 1 | 1 | 60.000 | 180.000 | | | |
| 2 | 2 | 0 | .000 | 180.000 | | | |
| 2 | 2 | 1 | .000 | 180.000 | | | |
| 2 | 2 | 2 | 180.000 | | | | |
| 5/2 | 1/2 | 1/2 | 46.378 | 106.852 | 180.000 | | |
| 5/2 | 3/2 | 1/2 | .000 | 78.463 | 180.000 | | |
| 5/2 | 3/2 | 3/2 | 53.130 | 180.000 | | | |
| 5/2 | 5/2 | 1/2 | .000 | 180.000 | | | |
| 5/2 | 5/2 | 3/2 | .000 | 180.000 | | | |
| 5/2 | 5/2 | 5/2 | 180.000 | | | | |
| 3 | 0 | 0 | 39.231 | 90.000 | 140.768 | | |
| 3 | 1 | 0 | .000 | 63.435 | 116.565 | 180.000 | |
| 3 | 1 | 1 | 40.977 | 95.066 | 180.000 | | |
| 3 | 2 | 0 | .000 | 90.000 | 180.000 | | |
| 3 | 2 | 1 | .000 | 70.529 | 180.000 | | |
| 3 | 2 | 2 | 48.190 | 180.000 | | | |
| 3 | 3 | 0 | .000 | 180.000 | | | |
| 3 | 3 | 1 | .000 | 180.000 | | | |
| 3 | 3 | 2 | .000 | 180.000 | | | |
| 3 | 3 | 3 | 180.000 | | | | |
| 7/2 | 1/2 | 1/2 | 34.631 | 79.568 | 125.122 | 180.000 | |
| 7/2 | 3/2 | 1/2 | .000 | 56.844 | 105.141 | 180.000 | |
| 7/2 | 3/2 | 3/2 | 37.115 | 85.576 | 180.000 | | |
| 7/2 | 5/2 | 1/2 | .000 | 81.787 | 180.000 | | |
| 7/2 | 5/2 | 3/2 | .000 | 64.623 | 180.000 | | |
| 7/2 | 5/2 | 5/2 | 44.415 | 180.000 | | | |
| 7/2 | 7/2 | 1/2 | .000 | 180.000 | | | |
| 7/2 | 7/2 | 3/2 | .000 | 180.000 | | | |
| 7/2 | 7/2 | 5/2 | .000 | 180.000 | | | |
| 7/2 | 7/2 | 7/2 | 180.000 | | | | |
| 4 | 0 | 0 | 30.556 | 70.124 | 109.876 | 149.444 | |
| 4 | 1 | 0 | .000 | 49.107 | 90.000 | 130.893 | 180.000 |
| 4 | 1 | 1 | 31.349 | 72.152 | 114.205 | 180.000 | |
| 4 | 2 | 0 | .000 | 67.792 | 112.208 | 180.000 | |
| 4 | 2 | 1 | .000 | 51.978 | 96.659 | 180.000 | |
| 4 | 2 | 2 | 34.175 | 80.057 | 180.000 | | |
| 4 | 3 | 0 | .000 | 90.000 | 180.000 | | |
| 4 | 3 | 1 | .000 | 75.523 | 180.000 | | |
| 4 | 3 | 2 | .000 | 60.000 | 180.000 | | |
| 4 | 3 | 3 | 41.410 | 180.000 | | | |
| 4 | 4 | 0 | .000 | 180.000 | | | |
| 4 | 4 | 1 | .000 | 180.000 | | | |
| 4 | 4 | 2 | .000 | 180.000 | | | |
| 4 | 4 | 3 | .000 | 180.000 | | | |
| 4 | 4 | 4 | 180.000 | | | | |
| 9/2 | 1/2 | 1/2 | 27.651 | 63.403 | 99.624 | 136.094 | 180.000 |
| 9/2 | 3/2 | 1/2 | .000 | 44.861 | 82.397 | 120.517 | 180.000 |
| 9/2 | 3/2 | 3/2 | 28.854 | 66.570 | 105.871 | 180.000 | |
| 9/2 | 5/2 | 1/2 | .000 | 62.453 | 103.901 | 180.000 | |
| 9/2 | 5/2 | 3/2 | .000 | 48.190 | 90.000 | 180.000 | |
| 9/2 | 5/2 | 5/2 | 31.840 | 74.836 | 180.000 | | |
| 9/2 | 7/2 | 1/2 | .000 | 83.621 | 180.000 | | |
| 9/2 | 7/2 | 3/2 | .000 | 70.529 | 180.000 | | |
| 9/2 | 7/2 | 5/2 | .000 | 56.251 | 180.000 | | |
| 9/2 | 7/2 | 7/2 | 38.942 | 180.000 | | | |
| 9/2 | 9/2 | 1/2 | .000 | 180.000 | | | |
| 9/2 | 9/2 | 3/2 | .000 | 180.000 | | | |
| 9/2 | 9/2 | 5/2 | .000 | 180.000 | | | |
| 9/2 | 9/2 | 7/2 | .000 | 180.000 | | | |
| 9/2 | 9/2 | 9/2 | 180.000 | | | | |

## TABLE. Zeros of the Reduced Rotation Matrix

| J | MF | MI | Angles of Zeros | | | | | | |
|---|----|----|-----|-----|-----|-----|-----|-----|-----|
| 5 | 0 | 0 | 25.017 | 57.420 | 90.000 | 122.579 | 154.983 | | |
| 5 | 1 | 0 | .000 | 40.088 | 73.427 | 106.573 | 139.912 | 180.000 | |
| 5 | 1 | 1 | 25.445 | 58.480 | 91.954 | 126.278 | 180.000 | | |
| 5 | 2 | 0 | .000 | 54.736 | 90.000 | 125.264 | 180.000 | | |
| 5 | 2 | 1 | .000 | 41.564 | 76.500 | 112.437 | 180.000 | | |
| 5 | 2 | 2 | 26.875 | 62.110 | 99.194 | 180.000 | | | |
| 5 | 3 | 0 | .000 | 70.529 | 109.471 | 180.000 | | | |
| 5 | 3 | 1 | .000 | 58.224 | 97.273 | 180.000 | | | |
| 5 | 3 | 2 | .000 | 45.130 | 84.578 | 180.000 | | | |
| 5 | 3 | 3 | 29.928 | 70.529 | 180.000 | | | | |
| 5 | 4 | 0 | .000 | 90.000 | 180.000 | | | | |
| 5 | 4 | 1 | .000 | 78.463 | 180.000 | | | | |
| 5 | 4 | 2 | .000 | 66.422 | 180.000 | | | | |
| 5 | 4 | 3 | .000 | 53.130 | 180.000 | | | | |
| 5 | 4 | 4 | 36.870 | 180.000 | | | | | |
| 5 | 5 | 0 | .000 | 180.000 | | | | | |
| 5 | 5 | 1 | .000 | 180.000 | | | | | |
| 5 | 5 | 2 | .000 | 180.000 | | | | | |
| 5 | 5 | 3 | .000 | 180.000 | | | | | |
| 5 | 5 | 4 | .000 | 180.000 | | | | | |
| 5 | 5 | 5 | 180.000 | | | | | | |
| 11/2 | 1/2 | 1/2 | 23.018 | 52.845 | 82.674 | 113.012 | 143.411 | 180.000 | |
| 11/2 | 3/2 | 1/2 | .000 | 37.127 | 68.069 | 98.996 | 130.600 | 180.000 | |
| 11/2 | 3/2 | 3/2 | 23.695 | 54.510 | 85.889 | 118.493 | 180.000 | | |
| 11/2 | 5/2 | 1/2 | .000 | 50.987 | 84.008 | 117.462 | 180.000 | | |
| 11/2 | 5/2 | 3/2 | .000 | 38.905 | 71.737 | 105.865 | 180.000 | | |
| 11/2 | 5/2 | 5/2 | 25.235 | 58.450 | 93.671 | 180.000 | | | |
| 11/2 | 7/2 | 1/2 | .000 | 66.057 | 102.945 | 180.000 | | | |
| 11/2 | 7/2 | 3/2 | .000 | 54.763 | 91.806 | 180.000 | | | |
| 11/2 | 7/2 | 5/2 | .000 | 42.590 | 80.045 | 180.000 | | | |
| 11/2 | 7/2 | 7/2 | 26.322 | 66.894 | 180.000 | | | | |
| 11/2 | 9/2 | 1/2 | .000 | 84.784 | 180.000 | | | | |
| 11/2 | 9/2 | 3/2 | .000 | 74.173 | 180.000 | | | | |
| 11/2 | 9/2 | 5/2 | .000 | 62.964 | 180.000 | | | | |
| 11/2 | 9/2 | 7/2 | .000 | 50.479 | 180.000 | | | | |
| 11/2 | 9/2 | 9/2 | 35.097 | 180.000 | | | | | |
| 11/2 | 11/2 | 1/2 | .000 | 180.000 | | | | | |
| 11/2 | 11/2 | 3/2 | .000 | 180.000 | | | | | |
| 11/2 | 11/2 | 5/2 | .000 | 180.000 | | | | | |
| 11/2 | 11/2 | 7/2 | .000 | 180.000 | | | | | |
| 11/2 | 11/2 | 9/2 | .000 | 180.000 | | | | | |
| 11/2 | 11/2 | 11/2 | 180.000 | | | | | | |
| 6 | 0 | 0 | 21.177 | 48.608 | 76.195 | 103.805 | 131.392 | 158.823 | |
| 6 | 1 | 0 | .000 | 33.878 | 62.040 | 90.000 | 117.960 | 146.122 | 180.000 |
| 6 | 1 | 1 | 21.433 | 49.229 | 77.276 | 105.585 | 134.596 | 180.000 | |
| 6 | 2 | 0 | .000 | 45.993 | 75.489 | 104.511 | 134.007 | 180.000 | |
| 6 | 2 | 1 | .000 | 34.743 | 63.760 | 92.917 | 123.113 | 180.000 | |
| 6 | 2 | 2 | 22.264 | 51.260 | 80.919 | 112.069 | 180.000 | | |
| 6 | 3 | 0 | .000 | 58.518 | 90.000 | 121.462 | 180.000 | | |
| 6 | 3 | 1 | .000 | 47.925 | 79.110 | 111.041 | 180.000 | | |
| 6 | 3 | 2 | .000 | 36.702 | 67.781 | 100.364 | 180.000 | | |
| 6 | 3 | 3 | 23.696 | 55.374 | 88.997 | 180.000 | | | |
| 6 | 4 | 0 | .000 | 72.452 | 107.548 | 180.000 | | | |
| 6 | 4 | 1 | .000 | 62.357 | 97.506 | 180.000 | | | |
| 6 | 4 | 2 | .000 | 51.859 | 87.188 | 180.000 | | | |
| 6 | 4 | 3 | .000 | 40.437 | 76.179 | 180.000 | | | |
| 6 | 4 | 4 | 26.950 | 63.773 | 180.000 | | | | |
| 6 | 5 | 0 | .000 | 90.000 | 180.000 | | | | |
| 6 | 5 | 1 | .000 | 80.406 | 180.000 | | | | |
| 6 | 5 | 2 | .000 | 70.529 | 180.000 | | | | |
| 6 | 5 | 3 | .000 | 60.000 | 180.000 | | | | |
| 6 | 5 | 4 | .000 | 48.190 | 180.000 | | | | |
| 6 | 5 | 5 | 33.557 | 180.000 | | | | | |
| 6 | 6 | 0 | .000 | 180.000 | | | | | |
| 6 | 6 | 1 | .000 | 180.000 | | | | | |
| 6 | 6 | 2 | .000 | 180.000 | | | | | |
| 6 | 6 | 3 | .000 | 180.000 | | | | | |
| 6 | 6 | 4 | .000 | 180.000 | | | | | |
| 6 | 6 | 5 | .000 | 180.000 | | | | | |
| 6 | 6 | 6 | 180.000 | | | | | | |

TABLE. Zeros of the Reduced Rotation Matrix

| J | MF | MI | Angles of Zeros | | | | | | | |
|---|----|----|---|---|---|---|---|---|---|---|
| 13/2 | 1/2 | 1/2 | 19.718 | 45.264 | 70.972 | 96.739 | 122.579 | 148.637 | 180.000 | |
| 13/2 | 3/2 | 1/2 | .000 | 31.696 | 58.073 | 84.324 | 110.723 | 137.742 | 180.000 | |
| 13/2 | 3/2 | 3/2 | 20.137 | 46.274 | 72.714 | 99.544 | 127.414 | 180.000 | | |
| 13/2 | 5/2 | 1/2 | .000 | 43.215 | 71.001 | 98.466 | 126.801 | 180.000 | | |
| 13/2 | 5/2 | 3/2 | .000 | 32.767 | 60.192 | 87.878 | 116.866 | 180.000 | | |
| 13/2 | 5/2 | 5/2 | 21.064 | 48.535 | 76.742 | 106.631 | 180.000 | | | |
| 13/2 | 7/2 | 1/2 | .000 | 55.200 | 85.052 | 115.226 | 180.000 | | | |
| 13/2 | 7/2 | 3/2 | .000 | 45.361 | 74.999 | 105.614 | 180.000 | | | |
| 13/2 | 7/2 | 5/2 | .000 | 34.836 | 64.423 | 95.663 | 180.000 | | | |
| 13/2 | 7/2 | 7/2 | 22.736 | 52.741 | 84.971 | 180.000 | | | | |
| 13/2 | 9/2 | 1/2 | .000 | 68.608 | 102.175 | 180.000 | | | | |
| 13/2 | 9/2 | 3/2 | .000 | 59.226 | 92.672 | 180.000 | | | | |
| 13/2 | 9/2 | 5/2 | .000 | 49.377 | 83.215 | 180.000 | | | | |
| 13/2 | 9/2 | 7/2 | .000 | 38.582 | 72.830 | 180.000 | | | | |
| 13/2 | 9/2 | 9/2 | 25.761 | 61.053 | 180.000 | | | | | |
| 13/2 | 11/2 | 1/2 | .000 | 85.588 | 180.000 | | | | | |
| 13/2 | 11/2 | 3/2 | .000 | 76.658 | 180.000 | | | | | |
| 13/2 | 11/2 | 5/2 | .000 | 67.380 | 180.000 | | | | | |
| 13/2 | 11/2 | 7/2 | .000 | 57.421 | 180.000 | | | | | |
| 13/2 | 11/2 | 9/2 | .000 | 46.187 | 180.000 | | | | | |
| 13/2 | 11/2 | 11/2 | 32.204 | 180.000 | | | | | | |
| 13/2 | 13/2 | 1/2 | .000 | 180.000 | | | | | | |
| 13/2 | 13/2 | 3/2 | .000 | 180.000 | | | | | | |
| 13/2 | 13/2 | 5/2 | .000 | 180.000 | | | | | | |
| 13/2 | 13/2 | 7/2 | .000 | 180.000 | | | | | | |
| 13/2 | 13/2 | 9/2 | .000 | 180.000 | | | | | | |
| 13/2 | 13/2 | 11/2 | .000 | 180.000 | | | | | | |
| 13/2 | 13/2 | 13/2 | 180.000 | | | | | | | |
| 7 | 0 | 0 | 18.358 | 42.138 | 66.056 | 90.000 | 113.944 | 137.862 | 161.642 | 180.000 |
| 7 | 1 | 0 | .000 | 29.338 | 53.722 | 77.919 | 102.081 | 126.278 | 150.662 | 180.000 |
| 7 | 1 | 1 | 18.524 | 42.534 | 66.726 | 91.036 | 115.554 | 140.679 | 180.000 | |
| 7 | 2 | 0 | .000 | 39.695 | 65.108 | 90.000 | 114.892 | 140.305 | 180.000 | |
| 7 | 2 | 1 | .000 | 29.890 | 54.793 | 79.644 | 104.773 | 130.844 | 180.000 | |
| 7 | 2 | 2 | 19.081 | 43.799 | 68.893 | 94.481 | 121.360 | 180.000 | | |
| 7 | 3 | 0 | .000 | 50.160 | 76.933 | 103.067 | 129.840 | 180.000 | | |
| 7 | 3 | 1 | .000 | 40.860 | 67.249 | 93.450 | 120.736 | 180.000 | | |
| 7 | 3 | 2 | .000 | 31.096 | 57.171 | 87.603 | 111.530 | 180.000 | | |
| 7 | 3 | 3 | 20.040 | 46.206 | 73.163 | 101.940 | 180.000 | | | |
| 7 | 4 | 0 | .000 | 61.289 | 90.000 | 118.710 | 180.000 | | | |
| 7 | 4 | 1 | .000 | 52.398 | 80.866 | 109.898 | 180.000 | | | |
| 7 | 4 | 2 | .000 | 43.173 | 71.461 | 100.939 | 180.000 | | | |
| 7 | 4 | 3 | .000 | 33.230 | 61.524 | 91.579 | 180.000 | | | |
| 7 | 4 | 4 | 21.730 | 50.453 | 81.452 | 180.000 | | | | |
| 7 | 5 | 0 | .000 | 73.698 | 106.102 | 180.000 | | | | |
| 7 | 5 | 1 | .000 | 65.332 | 97.565 | 180.000 | | | | |
| 7 | 5 | 2 | .000 | 56.530 | 88.658 | 180.000 | | | | |
| 7 | 5 | 3 | .000 | 47.222 | 79.748 | 180.000 | | | | |
| 7 | 5 | 4 | .000 | 36.962 | 69.890 | 180.000 | | | | |
| 7 | 5 | 5 | 24.716 | 58.656 | 180.000 | | | | | |
| 7 | 6 | 0 | .000 | 90.000 | 180.000 | | | | | |
| 7 | 6 | 1 | .000 | 81.787 | 180.000 | | | | | |
| 7 | 6 | 2 | .000 | 73.398 | 180.000 | | | | | |
| 7 | 6 | 3 | .000 | 64.623 | 180.000 | | | | | |
| 7 | 6 | 4 | .000 | 55.150 | 180.000 | | | | | |
| 7 | 6 | 5 | .000 | 44.415 | 180.000 | | | | | |
| 7 | 6 | 6 | 31.003 | 180.000 | | | | | | |
| 7 | 7 | 0 | .000 | 180.000 | | | | | | |
| 7 | 7 | 1 | .000 | 180.000 | | | | | | |
| 7 | 7 | 2 | .000 | 180.000 | | | | | | |
| 7 | 7 | 3 | .000 | 180.000 | | | | | | |
| 7 | 7 | 4 | .000 | 180.000 | | | | | | |
| 7 | 7 | 5 | .000 | 180.000 | | | | | | |
| 7 | 7 | 6 | .000 | 180.000 | | | | | | |
| 7 | 7 | 7 | 180.000 | | | | | | | |

TABLE. Zeros of the Reduced Rotation Matrix

| J | MF | MI | Angles of Zeros | | | | | | | | |
|---|----|----|---|---|---|---|---|---|---|---|---|
| 15/2 | 1/2 | 1/2 | 17.246 | 39.588 | 62.066 | 84.569 | 107.143 | 129.756 | 152.558 | 180.000 | |
| 15/2 | 3/2 | 1/2 | .000 | 27.654 | 50.669 | 73.528 | 96.415 | 119.466 | 143.071 | 180.000 | |
| 15/2 | 3/2 | 3/2 | 17.524 | 40.230 | 63.179 | 86.263 | 109.719 | 134.063 | 180.000 | | |
| 15/2 | 5/2 | 1/2 | .000 | 37.353 | 61.629 | 85.277 | 109.058 | 133.666 | 180.000 | | |
| 15/2 | 5/2 | 3/2 | .000 | 26.362 | 52.021 | 75.669 | 99.744 | 124.980 | 180.000 | | |
| 15/2 | 5/2 | 5/2 | 18.124 | 41.666 | 65.627 | 90.148 | 116.143 | 180.000 | | | |
| 15/2 | 7/2 | 1/2 | .000 | 47.597 | 73.074 | 98.066 | 123.953 | 180.000 | | | |
| 15/2 | 7/2 | 3/2 | .000 | 38.908 | 64.046 | 89.142 | 115.517 | 180.000 | | | |
| 15/2 | 7/2 | 5/2 | .000 | 29.658 | 54.569 | 79.912 | 106.692 | 180.000 | | | |
| 15/2 | 7/2 | 7/2 | 19.152 | 44.185 | 70.049 | 97.833 | 180.000 | | | | |
| 15/2 | 9/2 | 1/2 | .000 | 58.326 | 85.785 | 113.490 | 180.000 | | | | |
| 15/2 | 9/2 | 3/2 | .000 | 49.989 | 77.258 | 105.277 | 180.000 | | | | |
| 15/2 | 9/2 | 5/2 | .000 | 41.275 | 68.424 | 96.651 | 180.000 | | | | |
| 15/2 | 9/2 | 7/2 | .000 | 31.827 | 58.988 | 87.986 | 180.000 | | | | |
| 15/2 | 9/2 | 9/2 | 20.846 | 48.441 | 76.342 | 180.000 | | | | | |
| 15/2 | 11/2 | 1/2 | .000 | 70.529 | 101.537 | 180.000 | | | | | |
| 15/2 | 11/2 | 3/2 | .000 | 62.493 | 93.547 | 180.000 | | | | | |
| 15/2 | 11/2 | 5/2 | .000 | 54.175 | 85.333 | 180.000 | | | | | |
| 15/2 | 11/2 | 7/2 | .000 | 45.328 | 76.686 | 180.000 | | | | | |
| 15/2 | 11/2 | 9/2 | .000 | 35.530 | 67.262 | 180.000 | | | | | |
| 15/2 | 11/2 | 11/2 | 23.789 | 56.521 | 180.000 | | | | | | |
| 15/2 | 13/2 | 1/2 | .000 | 86.177 | 180.000 | | | | | | |
| 15/2 | 13/2 | 3/2 | .000 | 78.463 | 180.000 | | | | | | |
| 15/2 | 13/2 | 5/2 | .000 | 70.529 | 180.000 | | | | | | |
| 15/2 | 13/2 | 7/2 | .000 | 62.182 | 180.000 | | | | | | |
| 15/2 | 13/2 | 9/2 | .000 | 53.130 | 180.000 | | | | | | |
| 15/2 | 13/2 | 11/2 | .000 | 42.833 | 180.000 | | | | | | |
| 15/2 | 13/2 | 13/2 | 29.926 | 180.000 | | | | | | | |
| 15/2 | 15/2 | 1/2 | .000 | 180.000 | | | | | | | |
| 15/2 | 15/2 | 3/2 | .000 | 180.000 | | | | | | | |
| 15/2 | 15/2 | 5/2 | .000 | 180.000 | | | | | | | |
| 15/2 | 15/2 | 7/2 | .000 | 180.000 | | | | | | | |
| 15/2 | 15/2 | 9/2 | .000 | 180.000 | | | | | | | |
| 15/2 | 15/2 | 11/2 | .000 | 180.000 | | | | | | | |
| 15/2 | 15/2 | 13/2 | .000 | 180.000 | | | | | | | |
| 15/2 | 15/2 | 15/2 | 180.000 | | | | | | | | |
| 8 | 0 | 0 | 16.201 | 37.187 | 58.296 | 79.430 | 100.570 | 121.704 | 142.813 | 163.799 | 180.000 |
| 8 | 1 | 0 | .000 | 25.874 | 47.376 | 68.708 | 90.000 | 111.292 | 132.624 | 154.126 | 180.000 |
| 8 | 1 | 1 | 16.315 | 37.456 | 58.742 | 80.096 | 101.539 | 123.164 | 145.322 | 180.000 | |
| 8 | 2 | 0 | .000 | 34.931 | 57.275 | 79.124 | 100.876 | 122.725 | 145.069 | 180.000 | |
| 8 | 2 | 1 | .000 | 26.247 | 48.091 | 69.828 | 91.653 | 113.761 | 136.711 | 180.000 | |
| 8 | 2 | 2 | 16.670 | 38.301 | 60.154 | 82.233 | 104.744 | 128.394 | 180.000 | | |
| 8 | 3 | 0 | .000 | 43.955 | 67.339 | 90.000 | 112.661 | 136.045 | 180.000 | | |
| 8 | 3 | 1 | .000 | 35.726 | 58.665 | 81.256 | 104.089 | 127.984 | 180.000 | | |
| 8 | 3 | 2 | .000 | 27.048 | 49.635 | 72.284 | 95.407 | 119.691 | 180.000 | | |
| 8 | 3 | 3 | 17.320 | 39.853 | 62.794 | 86.378 | 111.575 | 180.000 | | | |
| 8 | 4 | 0 | .000 | 53.316 | 78.017 | 101.983 | 126.684 | 180.000 | | | |
| 8 | 4 | 1 | .000 | 45.396 | 69.758 | 93.762 | 118.855 | 180.000 | | | |
| 8 | 4 | 2 | .000 | 37.191 | 61.269 | 85.397 | 110.952 | 180.000 | | | |
| 8 | 4 | 3 | .000 | 28.403 | 52.295 | 76.679 | 102.805 | 180.000 | | | |
| 8 | 4 | 4 | 18.373 | 42.409 | 67.307 | 94.196 | 180.000 | | | | |
| 8 | 5 | 0 | .000 | 63.435 | 90.000 | 116.565 | 180.000 | | | | |
| 8 | 5 | 1 | .000 | 55.762 | 82.131 | 108.938 | 180.000 | | | | |
| 8 | 5 | 2 | .000 | 47.888 | 74.105 | 101.213 | 180.000 | | | | |
| 8 | 5 | 3 | .000 | 39.609 | 65.733 | 93.233 | 180.000 | | | | |
| 8 | 5 | 4 | .000 | 30.589 | 56.744 | 84.791 | 180.000 | | | | |
| 8 | 5 | 5 | 20.063 | 48.652 | 75.566 | 180.000 | | | | | |
| 8 | 6 | 0 | .000 | 75.037 | 104.963 | 180.000 | | | | | |
| 8 | 6 | 1 | .000 | 67.594 | 97.537 | 180.000 | | | | | |
| 8 | 6 | 2 | .000 | 60.000 | 90.000 | 180.000 | | | | | |
| 8 | 6 | 3 | .000 | 52.095 | 82.204 | 180.000 | | | | | |
| 8 | 6 | 4 | .000 | 43.847 | 73.955 | 180.000 | | | | | |
| 8 | 6 | 5 | .000 | 34.253 | 64.948 | 180.000 | | | | | |
| 8 | 6 | 6 | 22.959 | 54.004 | 180.000 | | | | | | |
| 8 | 7 | 0 | .000 | 90.000 | 180.000 | | | | | | |
| 8 | 7 | 1 | .000 | 82.819 | 180.000 | | | | | | |
| 8 | 7 | 2 | .000 | 73.522 | 180.000 | | | | | | |
| 8 | 7 | 3 | .000 | 67.976 | 180.000 | | | | | | |
| 8 | 7 | 4 | .000 | 60.000 | 180.000 | | | | | | |
| 8 | 7 | 5 | .000 | 51.318 | 180.000 | | | | | | |
| 8 | 7 | 6 | .000 | 41.410 | 180.000 | | | | | | |
| 8 | 7 | 7 | 28.955 | 180.000 | | | | | | | |
| 8 | 8 | 0 | .000 | 180.000 | | | | | | | |
| 8 | 8 | 1 | .000 | 180.000 | | | | | | | |
| 8 | 8 | 2 | .000 | 180.000 | | | | | | | |
| 8 | 8 | 3 | .000 | 180.000 | | | | | | | |
| 8 | 8 | 4 | .000 | 180.000 | | | | | | | |
| 8 | 8 | 5 | .000 | 180.000 | | | | | | | |
| 8 | 8 | 6 | .000 | 180.000 | | | | | | | |
| 8 | 8 | 7 | .000 | 180.000 | | | | | | | |
| 8 | 8 | 8 | 180.000 | | | | | | | | |