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ANGULAR CORRELATIONS AND NUCLEAR POLARIZATION FROM THE INELASTIC SCATTERING OF ALPHA PARTICLES

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Abstract: When alpha particles are inelastically scattered from an even nucleus, all particles in the interaction except the excited nucleus are spinless, with the results that the angular momentum relationships of the interaction are greatly simplified and that the polarization of the excited nucleus and the angular distribution of the decay radiation from this nucleus are related in a straight forward manner. For this situation, the functional dependence of the angular correlation between the decay radiation and the scattered alpha particle is calculated in terms of parameters describing the polarization of the excited nucleus, assuming either gamma-ray or alpha-particle decay of this nucleus. The angular dependence of the gamma-ray circular polarization is also calculated and related to the nuclear polarization. Experiments are outlined for measuring nuclear polarization with particle-gamma or particle-particle correlation studies, and the kinematical problems inherent in the latter are discussed.

1. Introduction

When a nuclear reaction occurs, the residual nucleus is polarized, i.e. its angular momentum substates are populated in a particular way determined by the mechanism and the angular momentum relationships of the reaction. If the residual nucleus subsequently decays by emitting some sort of radiation, its polarization will, in turn, determine the angular distribution of this radiation. It is, therefore, possible to obtain information about the polarization of a nucleus excited in a nuclear reaction by studying the angular correlation of its decay radiation with the reaction particles.

When an even nucleus is excited by the inelastic scattering of alpha particles the situation is particularly favourable for the study of nuclear polarization. Of the two initial and two final particles in the reaction, all but the excited nucleus are spinless, thereby greatly simplifying the angular momentum relationships. In this paper the functional dependence of the angular correlation between the scattered alpha and the decay radiation will be calculated, and the relation between this function and the polarization of the excited nucleus will be demonstrated, assuming either gamma-ray or alpha-particle emission by the excited nucleus. It will be shown that with a set of related correlation measurements it is experimentally feasible to determine completely the polarization of the excited nucleus following alpha-particle scattering.

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The theory of angular correlations has progressed from early work [1]), which used explicit summation over angular momentum substates, to the more elegant approach developed by Wigner, Racah, Blatt, Biedenharn, Rose, Fano and others [2]). This "new" formalism has had particular success facilitating the calculation of angular correlation functions where radioactive decay or compound-nucleus reactions are involved [3]), but it loses much of its elegant simplicity when applied to direct reaction processes. Most direct reaction calculations of this type have involved either simplifying assumptions [4,5]) or elaborate formal treatment and numerical evaluation [6,7]).

The approach of this paper will be to go back to the earlier method of explicitly summing over nuclear substates, while limiting the calculation to include primarily spinless particles so that the sums remain fairly simple. Attention will be directed to the decay of the polarized nucleus, while the reaction mechanism which induced this polarization will not, for the most part, be discussed here. However, it is expected that data on nuclear polarization will prove to be of considerable value in the study of reaction mechanisms.

The treatment which will be presented here differs from similar treatments of the decay of oriented nuclei [3,8]) in one important respect: such treatments generally assume that the nucleus has been oriented by external forces, e.g., an external magnetic field. In such a situation each nucleus will be in a definite and unique angular momentum substate, with a certain population of nuclei in each substate. In the situation of interest here, however, one must include the possibility that each nucleus can be in a *mixture* of substates, so that the angular distribution of the decay particles will be affected by coherent interference between these substates. Such coherent mixing of nuclear substates will generally be found in nuclei which are polarized by a nuclear reaction. In some particulars the same approach used in this paper has been taken by Schmidt [9]) in his work on proton spin-flip, by Alder and Winther [10]) in their calculations on nuclear alignment involving the impulse approximation at scattering angles near 180°, and by Litherland and Ferguson [11]) in their work demonstrating the use of the angular correlation measurement as a tool for nuclear spectroscopy. In the paper which follows this one [12]), the formalism which is developed here will be applied to α-γ angular correlation measurements on C^{12} and Mg^{24}, and information about the behaviour of the nuclear polarization as a function of scattering angle will be extracted from the data. A previous paper [13]) has already presented preliminary results of α-α correlation measurements involving the alpha-particle decay of the 9.6 MeV state in C^{12}.

2. The General Correlation Function

To represent the polarization of the excited nucleus, the irreducible spherical tensor p^j_m will be used. The mth element of this tensor is a complex number which characterizes the population of the mth substate of a nuclear level having total angular momentum j, the mth substate having spin projection m along the z or quantization axis. The polarization tensor is normalized so that $\sum_m |p^j_m|^2 = 1$, and its elements are essentially the same as the reduced amplitudes $Z^{\sigma_1 \sigma_2}_{j\zeta}$ of Goldfarb and Bromley [14]).

The quantization direction will be chosen as the direction $\boldsymbol{k}_i \times \boldsymbol{k}_f$, which is perpendicular to the reaction plane defined by the momenta of the incident and outgoing alpha particles. The beam direction $\boldsymbol{k}_i$ will be chosen as the x-axis. This coordinate system is shown in fig. 1. While this coordinate system represents a departure from many previous calculations [4-7]) (which generally quantize along some recoil direction in the reaction plane), it has several advantages. First, it is stationary, and does not change with the particle angle or beam energy; secondly, when this quantization axis is chosen the population of certain substates may be excluded by the reflectional

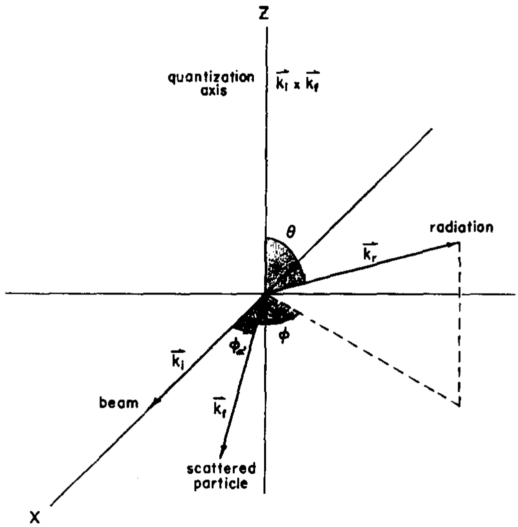

Fig. 1. Coordinate system used in the calculations. Directions $\boldsymbol{k}_i$, $\boldsymbol{k}_f$, and $\boldsymbol{k}_r$ of the beam, scattered particle, and emitted radiation, respectively, are indicated. The x-axis is taken as the direction $\boldsymbol{k}_i$; $\boldsymbol{k}_f$ lies in the x-y-plane; the Z or quantization axis is taken as the direction of $\boldsymbol{k}_i \times \boldsymbol{k}_f$. The angle ϕ_α, specifies the angle of scattering of the particle; θ and ϕ are the polar and azimuthal angles which specify the direction $\boldsymbol{k}_r$ of the emitted radiation.

symmetry theorem of A. Bohr [15]); and finally, any pure substate with this quantization axis will give an isotropic correlation pattern in the reaction plane, and conversely, an observed non-isotropic correlation pattern can be directly attributed to coherent interference between substates.

The Bohr theorem [15]) mentioned above requires that if a natural parity state (0^+, 1^-, 2^+, . . .) is excited in an even nucleus by inelastic alpha-particle scattering, only states will be excited for which $(J-m)$ is even. Thus, of the $2J+1$ possible substates, only $J+1$ will be excited. Moreover, of the $J+1$ non-zero coefficients in the polarization tensor which correspond to these substates, only J will be independent, since the overall phase of the polarization tensor $\boldsymbol{p}$ is arbitrary and the overall amplitude is restricted by the normalization condition. Thus, J phases and J amplitudes must be extracted from the correlation measurement to characterize the polarization of an excited nucleus having angular momentum J. (For the excitation of unnatural parity states[25]), $(J-m)$ will be odd, and J substates will be populated with $J-1$ independent. However, since these states usually decay by gamma *cascades*, their correlations are not covered by this calculation).

The transition amplitude for particle emission from a polarized nucleus with angular momentum J and characterized by a coherent mixture of substates m, which decays to a state of angular momentum J' and substates m', can be written †

$$A^{\sigma}_{JJ'}(\boldsymbol{k}) = \sum_{mm'} p^J_m \langle J'm'|H_\sigma(LM)|Jm\rangle. \tag{1}$$

Here p^J_m is an element of the polarization tensor; σ is the polarization of the decay particle, i.e., the projection of its spin along $\boldsymbol{k}$, its direction of propagation; and $H_\sigma(LM)$ is the Hamiltonian for emission of a particle with polarization σ and orbital angular momentum characterized by L and M. Note that $m = m'+M$, so a sum over M is unnecessary.

This expression is simplified by assuming that the excited nucleus decays to the ground state of an even nucleus, so that $J' = m' = 0$. In this case the sum over m' can be dropped and (1) becomes:

$$A^{\sigma}_{J}(\boldsymbol{k}) = \sum_{m} p^J_m \langle 00|H_\sigma(Jm)|Jm\rangle. \tag{2}$$

The Hamiltonian can be separated into an irreducible tensor operator $T(Jm)$ which is a function only of the internal coordinates of the nucleus, and the rotation operator [8]) $\boldsymbol{D}$ which describes the emission of the decay particle in terms of the external coordinate system (see Appendix A). Further, we may apply the Wigner-Eckart theorem [8]) to the matrix element involving $T(Jm)$ to reduce it to a Clebsch-Gordan coefficient and a reduced matrix element which does not depend on m. When this is done the transition amplitude becomes

$$\begin{aligned} A^{\sigma}_{J}(\boldsymbol{k}) &= \langle 0||T(J)||J\rangle \sum_{m} p^J_m D^J_{m\sigma}(\phi, \theta, 0) C(JJ0; m, -m) \\ &\propto \sum_{m} (-1)^m p^J_m e^{-im\phi} d^J_{m\sigma}(\theta). \end{aligned} \tag{3}$$

The correlation function which corresponds to this transition amplitude is

$$W^{\sigma}_{J}(\theta, \phi) = |A^{\sigma}_{J}(\boldsymbol{k})|^2 \propto |\sum_{m}(-1)^m p^J_m e^{-im\phi} d^J_{m\sigma}(\theta)|^2 \tag{4}$$

and, when the radiation from the decaying state is detected by an instrument which is not sensitive to the polarization of the emitted radiation, as is generally the case, the unpolarized correlation function is of interest. This function is obtained by summing over σ:

$$W_J(\theta, \phi) \propto \sum_{\sigma} | \sum_{m} (-1)^m p^J_m e^{-im\phi} d^J_{m\sigma}(\theta)|^2. \tag{5}$$

† Eq. (1) is generally valid, and may be applied in any case were J' is the excited state of a nucleus. The special case where J' is a nuclear ground state, however, requires special consideration. Here the sum over m' must be incoherent and should be applied after the amplitude has been squared, as in eq. (4). These considerations do not affect the present calculation, since $J' = 0$ and the sum over m' has been dropped.

Up to this point, the treatment has been fairly general, except for the assumption that the decay proceeds to a spinless final nucleus. This assumption will be valid in the majority of practical situations and is essential for an unambiguous determination of polarization of the excited nucleus. The correlation function (5) has been reduced to a fairly simple form involving the reduced rotation matrix elements $d^j_{m'm}(\theta)$. Appendix A contains a summary of the properties of this function.

As yet, however, no assumptions about the properties of the emitted particle have been made and no use has been made of the particular choice of the quantization axis which was mentioned in sect. 1 above. In the succeeding sections, this basic expression will be applied to correlations involving gamma rays and alpha particles, and the implications of expression (4) for gamma-ray polarization measurements will be investigated.

3. Gamma-Ray Angular Correlations

Gamma rays may be produced in states of right or left circular polarization, or in some coherent mixture of these corresponding to linear polarization. In the notation used here right and left circular polarization are designated by $\sigma = -1$ and 1, respectively, in accordance with the optical convention. Both of these polarization states must be included in the calculation of the correlation function, and so the rotation matrix elements $d^J_{m\,-1}$ and d^J_{m1} appear in the calculation and must be evaluated.

Since the first excited states of a large majority of the even nuclei have spin and parity 2^+, the calculation will be limited to the case $J = 2$. However, the methods outlined here can easily be applied to correlations involving other values of J, and a tabulation of values of rotation matrix elements up to $J = 4$ is included in appendix A for this purpose.

As mentioned above, excitation of a natural parity state by inelastic alpha-particle scattering will only produce substates in the excited nucleus such that $(J-m)$ is even. Thus, for $J = 2$ only $m = 0$ and ± 2 substates are populated and need be considered in the calculation. The corresponding elements of the rotation matrix, as given in appendix A, are

$$\begin{aligned} d^2_{2\pm1}(\theta) &= -\tfrac{1}{2}\sin\theta(1\pm\cos\theta), \\ d^2_{0\pm1}(\theta) &= \pm\tfrac{1}{2}\sqrt{6}\sin\theta\cos\theta, \\ d^2_{-2\pm1}(\theta) &= \tfrac{1}{2}\sin\theta(1\mp\cos\theta). \end{aligned} \tag{6}$$

Substituting these values into (3), the transition amplitude is

$$A^{\pm1}_2(\theta,\phi) \propto \sin\theta[p^2_2(1\pm\cos\theta)e^{-2i\phi}\mp 6p^2_0\cos\theta - p^2_{-2}(1\mp\cos\theta)e^{2i\phi}]. \tag{7}$$

The terms of the polarization tensor will be normalized to the $m = -2$ substate, thereby permitting the nuclear polarization to be characterized by two relative amplitudes a_2 and a_0, and two relative phases δ_2 and δ_0. These parameters are defined by the relations

$$\begin{aligned} p^2_2 &= a_2 e^{i\delta_2}p^2_{-2}, & a_2 &= |p^2_2/p^2_{-2}|, & e^{i\delta_2} &= p^2_2/a_2 p^2_{-2}, \\ p^2_0 &= a_0 e^{i\delta_0}p^2_{-2}, & a_0 &= |p^2_0/p^2_{-2}|, & e^{i\delta_0} &= p^2_0/a_0 p^2_{-2}. \end{aligned}$$

Using these normalized coefficients and eq. (7) above, and substituting for the first equality in eq. (4), the correlation function is

$$\begin{aligned} W_2^{\pm 1}(\theta, \phi) \propto \sin^2\theta\{&[a_2^2(1+\cos^2\theta \pm 2\cos\theta)+6a_0^2\cos^2\theta+(1+\cos^2\theta \mp 2\cos\theta)] \\ &-2\sqrt{6}a_0\cos\theta[a_2(\cos\theta \pm 1)\cos 2(\phi-\tfrac{1}{2}(\delta_2-\delta_0))+(\cos\theta \mp 1)\cos 2(\phi-\tfrac{1}{2}\delta_0)] \\ &-2a_2\sin^2\theta\cos 4(\phi-\tfrac{1}{4}\delta_2)\}. \end{aligned} \tag{8}$$

A sum over gamma-ray polarization removes the terms preceded by $\pm$ signs and thus the polarization-independent correlation function is

$$\begin{aligned} W_2(\theta, \phi) \propto \sin^2\theta\{&[(a_2^2+1)(1+\cos^2\theta)+6a_0^2\cos^2\theta] \\ &-2\sqrt{6}a_0\cos^2\theta[a_2\cos 2(\phi-\tfrac{1}{2}(\delta_2-\delta_0))+\cos 2(\phi-\tfrac{1}{2}\delta_0)] \\ &-2a_2\sin^2\theta\cos 4(\phi-\tfrac{1}{4}\delta_2)\}. \end{aligned} \tag{9}$$

In the reaction plane ($\theta = \frac{1}{2}\pi$) the terms which are dependent on a_0 drop out and the correlation function simplifies to

$$\begin{aligned} W_2(\tfrac{1}{2}\pi, \phi) &\propto (1+a_2^2)-2a_2\cos 4(\phi-\tfrac{1}{4}\delta_2), \\ &\propto (1-a_2)^2+4a_2\sin^2 2(\phi-\tfrac{1}{4}\delta_2). \end{aligned} \tag{10}$$

This for (10) is essentially that given by Schmidt [9]) for the correlation function in the reaction plane for protons, and he has pointed out [16]) that since

$$\begin{aligned} \int_0^\pi W_2(\tfrac{1}{2}\pi, \phi)\mathrm{d}\phi &\propto [1+a_2^2], \\ &\propto |p_2^2|^2+|p_{-2}^2|^2, \\ &\propto 1-|p_0^2|^2, \end{aligned}$$

one can, in principle, deduce the value of $|p_0^2|$ from the average gamma-ray intensity in the reaction plane. In practice, however, required information about cross sections, target thicknesses, counter efficiencies, and coincidence efficiency is generally subject to sizable errors. Moreover, the phase of p_0^2 is not measurable by such an approach.

The last expression in (10) is just that usually given for $J = 2$ correlations in the reaction plane [4-7,17]) i.e., $[A+B\sin_2^2(\phi-\phi_0)]$. In the correlation experiments the parameters ϕ_0 and A/B are usually extracted from the data and presented as a function of alpha-particle angle. From (10) the significance of these parameters in terms of nuclear polarization becomes apparent.

First, aside from the overall intensity mentioned above, the correlation pattern in the reaction plane is independent of p_0^2 and depends only on p_2^2 and p_{-2}^2. Moreover, aside from these parameters and the assumption that $J = 2$, the form of the correlation pattern is purely the result of the geometry of angular momentum and gives no information about the reaction itself. Whether the scattering is a direct reaction or the result of compound nucleus formation, the correlation will have form (10) in the reaction plane.

The relative phase δ_2 of the $m = \pm 2$ substates determines the angular position of the maxima and minima in the correlation pattern, and the relative amplitude a_2 determines the degree of isotropy which is shown by the correlation pattern. If $a_2 = 0$, indicating that the nucleus has no $m = +2$ component, the radiation pattern will have no ϕ dependence and will be completely isotropic. If $a_2 = 1$, so that the $m = \pm 2$ substates are populated equally, the correlation pattern will have no isotropic component and will be a pure sinusoid of the form $\sin^2 2(\phi - \frac{1}{4}\delta_2)$.

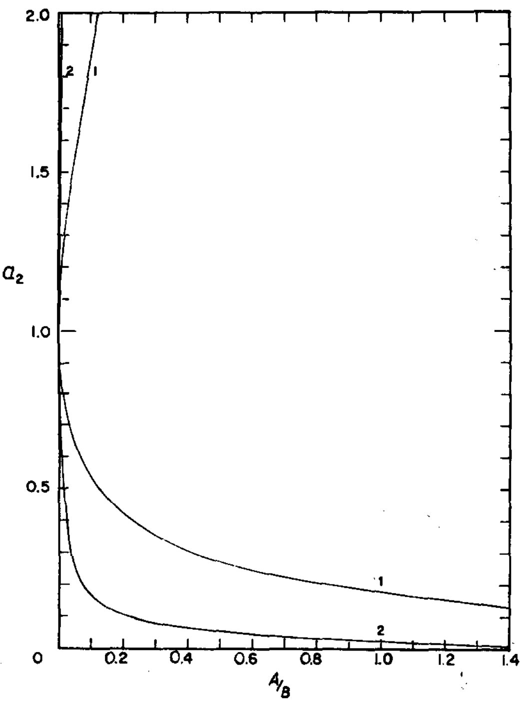

Fig. 2. Nuclear polarization parameters a_2 versus A/B ratio. Curve 1 represents the functional dependence of $a_2(A/B)$ on the scales as given. In curve 2 the A/B value is 10 times that given by the lower scale.

The functional relationship between A/B and a_2, as given by (10), contains implications of some importance concerning the information obtainable from correlation data. Setting $A/B = (1-a_2)^2/4a_2$ and solving for a_2 gives

$$a_2 = 1 - 2/(1 \pm \sqrt{1+B/A}). \tag{11}$$

The $\pm$ sign on the radical is of particular significance, for if the positive root is taken as a_2, it is found that the negative root corresponds to $1/a_2$. In other words, A/B has the same value for a_2 as for $1/a_2$. Fig. 2 illustrates the functional behaviour of a_2 as a function of A/B. The physical significance of this behaviour is that the observed correlation pattern for unpolarized radiation is unchanged by an exchange of positive and negative m-values. This property of the correlation is true not only in the reaction plane but over the whole sphere.

This puts a peculiar restriction on any correlation measurements which are insensitive to the polarization of the radiation; the ratio of positive to negative substate population can be measured, but it is not possible to determine which is which. It will be shown in the next section, however, that a fairly simple polarization determination removes this ambiguity.

As fig. 2 illustrates, a_2 has an extremely steep slope when A/B is near zero, and there will necessarily be fairly sizable errors in a_2 when is has values near 1. Moreover, unless there is a sizable difference between the contributions from the $m = 2$ and -2 substates, the isotropic component of the correlation pattern will be fairly small.

Of the four independent parameters, a_0, δ_0, a_2 and δ_2 which characterize the alignment of the excited nucleus, only two, or three at most, can be measured in the reaction plane. On the other hand, if a value of θ other than $\frac{1}{2}\pi$ is chosen for the angle of measurement, all four parameters can, in principle, be extracted from the observed correlation function but, in practice, the parameters are mixed in such a way as to make such an analysis extremely difficult. If, however, the correlation function is measured both in the reaction plane and at some angle to the plane, the parameters a_2 and δ_2 can be determined from the reaction plane data and used to analyse the more complex data measured outside the plane. This analysis proceeds as follows: consider the correlation function which will appear at $\theta = \frac{1}{4}\pi$. From (9) the correlation function is

$$W_2(\tfrac{1}{4}\pi, \phi) = b_1 + b_2 \cos 2\phi + b_3 \sin 2\phi + b_0(a_2 \cos 4(\phi - \tfrac{1}{4}\delta_2)), \tag{12}$$

where b_0 = arbitrary normalization constant

$$\begin{aligned} b_1 &= -\tfrac{3}{2}\, b_0(1 + a_2^2 + 2a_0^2), \\ b_2 &= +a_0 b_0 \sqrt{6}[\cos \delta_0 + a_2 \cos(\delta_2 - \delta_0)], \\ b_3 &= +a_0 b_0 \sqrt{6}[\sin \delta_0 + a_2 \cos(\delta_3 - \delta_0)]. \end{aligned}$$

The correlation function has been written in this form to simplify extraction of the b coefficients e.g., by least-squares analysis. Notice that the known parameters a_2 and δ_2 were grouped in the last term to find b_0, the normalization constant. The b coefficients are related to a_0 and δ_0 by the equations

$$\begin{aligned} \delta_0 &= \text{arctg}\left[\frac{b_3(1 + a_2 \cos \delta_2) + b_2 a_2 \sin \delta_2}{-b_2(1 + a_2 \cos \delta_2) + b_3 a_2 \sin \delta_2}\right], \\ a_0 &= b_2/[b_0\sqrt{6}(\cos \delta_0 + a_2 \cos(\delta_2 - \delta_0))]. \end{aligned} \tag{13}$$

Thus the correlation functions measured in and out of the reaction plane allow complete determination of the polarization parameters, aside from the m-value sign ambiguity mentioned above. Using the normalization condition that $p_{-2}^2 = (a_2^2 + a_0^2 + 1)^{-\frac{1}{2}}$, the polarization tensor p_m^2 can be represented as a column matrix by

$$p_m^2 = (a_2^2+a_0^2+1)^{-\frac{1}{2}} \begin{pmatrix} a_2 e^{i\delta_2} \\ 0 \\ a_0 e^{i\delta_0} \\ 0 \\ 1 \end{pmatrix} \tag{14}$$

4. Gamma-Ray Polarization and the Sign of *m*

As mentioned in sect. 3, a correlation function which is measured with a gamma-ray detector insensitive to gamma-ray polarization will be invariant under an exchange of positive and negative m values; there remains an overall sign ambiguity in any nuclear polarization measurement which depends upon angular correlation studies alone. However, if the gamma detector were made sensitive to gamma-ray polarization the ambiguity would not exist, since the correlation function of polarized gamma rays, as given by eq. (5), is strongly dependent on the sign of the m values.

While it is not experimentally feasible to measure the correlation of polarized gamma rays, it should be possible to measure the gamma ray polarization at a few angles to resolve the sign ambiguity, and such a measurement is simplified by the fact that only the sign, not the magnitude, of the gamma-ray polarization is required. The value of such polarization measurements as a test of direct reaction theory has already been discussed by Satchler [18]).

The gamma ray polarization function $S_J(\theta, \phi)$ is related to the correlation function by the usual relation

$$S_J(\theta, \phi) = \frac{W_J^{+1}(\theta, \phi) - W_J^{-1}(\theta, \phi)}{W_J^{+1}(\theta, \phi) + W_J^{-1}(\theta, \phi)} = \frac{W_J^{+1}(\theta, \phi) - W_J^{-1}(\theta, \phi)}{2W_J(\theta, \phi)}. \tag{15a}$$

Clearly, the denominator of this expression is just the polarization-independent correlation function.

For the case $J = 2$ the polarization function can be calculated from eq. (8) to give

$$S_2(\theta, \phi) = \frac{2\sin^2\theta\cos\theta}{W_2(\theta, \phi)} \{(a_2^2-1) - a_0\sqrt{6}[a_2\cos 2(\phi - \tfrac{1}{2}(\delta_2-\delta_0)) - \cos 2(\phi - \tfrac{1}{2}\delta_0)]\}. \tag{15b}$$

Analysis of existing experimental data [12]) shows that a_2 and presumably a_0 can be expected to vary fairly regularly between one and zero and out of phase with the angular distribution of inelastic scattering. If the assumption is made that the substate population must vary smoothly as a function of angle, then the dominance of one or the other of the $m = \pm 2$ substates can only change when $a_2 = 1$. Thus the gamma-ray polarization measurements need only be done at a few angles to remove the sign ambiguity.

Examination of eq. (15b) shows that the radiation will be unpolarized in the reaction plane but will become strongly polarized for values of θ near zero or π. Unfortunately, due to the overall $\sin^2\theta$ dependence of $W_2(\theta, \phi)$, the intensity of the radiation decreases just where it becomes strongly polarized. However, if one chooses some fairly shallow angle, e.g. $\frac{1}{8}\pi$, and also chooses a reaction particle angle such that a_0 and a_2 have small values, say 0.1 and an angle ϕ such that $\phi = +\frac{1}{4}\delta_2$, then we find that $S_2 = -0.66$. Here the gamma rays are polarized with enough intensity to make possible a measurement of the sign of their polarization.

Thus at least for $J = 2$, nuclear polarization can be *completely* determined as a function of scattering angle from the correlation pattern and polarization of the gamma radiation emitted by the excited nucleus. In the next section it will be shown that the same information can be obtained when the nucleus decays by alpha-particle emission, although in that case the sign ambiguity cannot be resolved.

5. Angular Correlations for Alpha Decay

When the nucleus which has been excited by alpha-particle inelastic scattering breaks up by alpha-particle emission, the correlation function will differ from that calculated above because of the difference in spin of the photon and alpha particle. Because the alpha particle is spinless, calculation of the correlation function is simplified somewhat since the sum over polarization states of the emitted radiation is removed. Consequently, however, it is not longer possible to resolve by a polarization measurement the sign ambiguity discussed in sect. 4.

The chief disadvantage of alpha-alpha over alpha-gamma correlations is the increased complexity of the kinematical relationships arising from the effect of the momentum of the decay alpha as measured in the laboratory system of the velocity of the recoiling excited nucleus. This creates serious experimental problems, particularly in the lighter nuclei. Since the correlation functions calculated here assume that the decaying nucleus is at rest, the angular coordinates used in the calculation are, in effect, measured in the centre of mass of the recoiling excited nucleus, hereafter denoted as the RCM system. Thus the transformations from the laboratory to the RCM system and back must be well understood before the results of alpha-alpha angular correlation experiments can be interpreted in terms of this formalism. This kinematics problem has been dealt with to some extent in a previous paper [13]), and is discussed in greater detail in appendix B.

Despite these kinematical difficulties, there are several advantages of alpha-alpha over alpha-gamma angular correlation measurements:

(1) problems arising from background are greatly reduced;

(2) radiation from the decays of different states can be better resolved and separated, allowing simultaneous measurement of several correlation functions;

(3) counter solid angles can be restricted to minimize distortions due to finite geometry;

(4) counter efficiency will generally be 100 %.

For these reasons, alpha-alpha correlations are an excellent way of studying the polarization of excited states which are unbound to alpha decay.

In this calculation as in the gamma-ray case, only $J = 2$ will be explicitly calculated, but this serves as an example for extending this type of calculation to other J values. As before, the Bohr theorem excludes the population of the $m = 1$ and -1 substates. Therefore, only the rotation matrix elements d^2_{00} and $d^2_{\pm 20}$ are needed for the calculation. From the relations given in appendix A, the elements have the values

$$d^2_{\pm 20}(\theta) = \tfrac{1}{2}\sqrt{\tfrac{3}{2}}\sin^2\theta, \qquad d^2_{00}(\theta) = \tfrac{1}{2}(3\cos^2\theta - 1).$$

Substituting these values into (3), taking the absolute square of the transition amplitude, and normalizing the elements of the polarization tensor to p^2_{-2} as before, give the correlation function

$$\begin{aligned} W^0_2(\theta, \phi) \propto {} & [\tfrac{3}{2}\sin^4\theta(a_2^2+1)+a_0^2(3\cos^2\theta-1)^2] \\ & +2\sqrt{\tfrac{3}{2}}a_0\sin^2\theta(3\cos^2\theta-1)[a_2\cos 2(\phi-\tfrac{1}{2}(\delta_2-\delta_0))+\cos 2(\phi-\tfrac{1}{2}\delta_0)] \\ & +3a_2\sin^4\theta\cos 4(\phi-\tfrac{1}{4}\delta_2). \end{aligned} \tag{16}$$

Unlike the gamma-ray correlation function (9), this expression does not reduce to a simple form in the reaction plane. On the other hand, when $\cos\theta_0 = \pm 1/\sqrt{3}$, at angles of approximately 55° and 125°, the correlation function does reduce to a simple form, because at these angles the rotation matrix element d^2_{00} vanishes. When this happens, the correlation function becomes

$$\begin{aligned} W^0_2(\theta_0, \phi) & \propto (1+a_2^2)+2a_2\cos 4(\phi-\tfrac{1}{4}\delta_2), \\ & \propto (1-a_2)^2+4a_2\cos^2 2(\phi-\tfrac{1}{4}\delta_2). \end{aligned} \tag{17}$$

This function is quite similar to (10), the gamma ray correlation function in the reaction plane, and differs only in phase from that expression. Because of this similarity, the remarks in sect. 3 concerning the relation of a_2 to the A/B ratio, the observed correlation pattern, and the substate sign ambiguity apply here also.

In the reaction plane all populated substates will contribute to the correlation function, which has the form

$$W^0_2(\tfrac{1}{2}\pi, \phi) = c_1+c_2\cos 2\phi+c_3\sin 2\phi+c_0(3a_2\cos 4(\phi-\tfrac{1}{4}\delta_2)),$$

where c_0 is an arbitrary normalization constant

$$\begin{aligned} c_1 &= c_0(\tfrac{3}{4}(1+a_2^2)+a_0^2), \\ c_2 &= -2\sqrt{\tfrac{3}{2}}a_0 c_0(\cos\delta_0+a_2\cos(\delta_2-\delta_0)), \\ c_3 &= -2\sqrt{\tfrac{3}{2}}a_0 c_0(\sin\delta_0+a_2\sin(\delta_2-\delta_0)). \end{aligned} \tag{18}$$

Here, as in (12), the function has been written in a form which will simplify extraction of the c coefficients. These coefficients are related to the parameters a_0 and δ_0 by

$$\begin{aligned} \delta_0 &= \mathrm{arctg}\left[\frac{c_3(1+a_2\cos\delta_2)+c_2(a_2\sin\delta_2)}{c_2(1+a_2\cos\delta_2)-c_3(a_2\sin\delta_2)}\right], \\ a_0 &= -c_2/[2c_0\sqrt{\tfrac{3}{2}}(\cos\delta_0+a_2\cos(\delta_2-\delta_0))]. \end{aligned} \tag{19}$$

Thus, by choosing the angles at which the correlation function was measured, it has again been possible to extract all the coefficients in the polarization matrix.

Since the rotation matrix has the property that $d^j_{m0} = (-1)^m d^j_{-m0}$, when one of the matrix elements vanishes, there will always be a significant simplification in the correlation function. Moreover, these favourable angles for correlation measurements will not, in general, correspond to the reaction plane (see appendix 1, subsects. A.1 and A.2). As an example the $J = 3$ correlation function [13]) is quite complicated at most angles, but when $\cos\theta = \pm 1/\sqrt{5}$, i.e., at 63° and 117°, $d^3_{\pm 10}$ vanishes and the correlation function becomes $W^0_3(\theta_0, \phi) \propto [(1+a^2_3)-2a_3 \cos 6(\phi-\frac{1}{6}\delta_3)]$, in perfect analogy with (10). States of higher angular momentum will also have such favourable angles, although the simplification will not always be so dramatic. Since it is important to understand the behaviour of the correlation function before undertaking a measurement of this type, subsect. A.5 of appendix A tabulates the zeroes of the relevant rotation matrix elements up to $J = 4$.

6. Conclusion

The calculations presented above demonstrate the relation between an observed correlation function and the polarization of the excited nucleus. Moreover, it has been shown that in many situations it is possible to determine completely the polarization of the excited nucleus.

While this treatment has been restricted to the scattering of alpha particles, it can be generalized to particles with spin by allowing the population of all the substates, including those for which $(J-m)$ is odd. On the other hand, the assumption that the excited nucleus is even and that the decay proceeds to a spinless ground state are fairly essential if useful information about nuclear polarization is to be obtained. Without these assumptions the condition of equality between the angular momentum and polarization of the excited nucleus and that of the decay transition is lost, and the correlation function is complicated by other effects. Correlations measured under these circumstances would be extremely difficult to interpret.

Thus far, the scope of this paper has been limited to those features of angular correlation functions which are produced by the polarization of the excited nucleus and the geometry of angular momentum, and has dealt primarily with the extraction of nuclear polarization information from angular correlation data. To put the preceding discussion in proper perspective, however, this polarization information should be related directly to the nuclear reaction which produced it and to the other measurable quantities which permit the study of the reaction.

The general reaction may be characterized in terms of a set of reaction amplitudes $T^{j'm's'\sigma'}_{jms\sigma}(\theta, E)$, where the primed symbols refer to the initial system and the unprimed symbols to the final system; j and m are the angular momentum and its projection for the initial and final nuclei, s and σ are the spin and its projection for the bombarding and reaction particles, and θ and E are the angle at which the reaction particle is observed and the energy of the bombarding particle. There will be $(2j+1)(2s+1)$ $(2j'+1)(2s'+1)$ such amplitudes, which may vary with angle and energy, and, except for symmetries and selection rules dictated by the details of the reaction, will be independent of each other. This set of amplitudes represents the maximum information which can be derived from the experimental study of a given reaction.

One can imagine an idealized experiment in which polarized beams, polarized targets, and polarization-sensitive detectors are used. In such an experiment it would be possible to reconstruct the reaction amplitudes completely (aside from an overall phase), provided as many independent experimental parameters were determined as the number of independent amplitudes. Such determinations might include, however, such formidable measurements as beam- and target-polarization-dependent cross sections, nuclear polarizations, and reaction-particle polarizations.

Clearly, such difficult measurements are impractical for most reactions, but for the special case of the inelastic scattering of alpha particles from even nuclei, measurement of the differential cross section and the nuclear polarization tensor, as defined in sect. 2 of this paper, permits unambiguous reconstruction of the complete set of amplitudes. This is possible because in this case $j' = s' = s = 0$, and so there are at most $2j+1$ independent amplitudes to be determined. The polarization tensor, which has $2j+1$ elements, together with the cross section, will in all cases provide just the number of independent quantities required for the reconstruction of the amplitudes within an overall phase.

The cross section is related to the scattering amplitudes by the relation $\mathrm{d}\sigma/\mathrm{d}\Omega = |\sum_m T_{jm}|^2 = \sum_m |T_{jm}|^2$, with the sum taken outside the square in the last expression because the cross terms vanish when averaged over nuclear coordinates. The elements of the polarization tensor are related to the scattering amplitudes by $p^j_m = T_{jm}/\sqrt{(\sum_m |T_{jm}|^2)}$. Thus measurements of the cross section and nuclear polarization are not redundant, but give completmentary information about the nuclear reaction, for the cross section is independent of the relative phases and relative magnitudes of the amplitudes and depends only on their overall absolute magnitude, while the polarization tensor is independent of this overall magnitude but gives directly the relative phases and relative magnitudes of the scattering amplitudes. In terms of the differential cross section and polarization tensor, the scattering amplitudes are given within an overall phase by $T_{jm} = p^j_m \sqrt{(\mathrm{d}\sigma/\mathrm{d}\Omega)}$. (While it is not possible to determine this overall phase, it has recently been shown [27]) that the overall phase *difference* between amplitudes at slightly different energies can, in certain cases, be determined from a study of nuclear bremsstrahlung.)

It is clear that the determination of scattering amplitudes in this way can give improved insight in the study of nuclear reactions and will provide more rigorous comparisons of experiment with theoretical predictions. This type of test should be all the more interesting because compound nucleus [26]) and direct reaction theories [4, 5, 7]) involving different degrees of approximation apparently make very different predictions concerning the scattering amplitudes while predicting similar behaviour of the differential cross section.

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Appendix A

PROPERTIES OF THE ROTATION MATRIX

In the above calculations, the transition amplitudes and correlation functions were written in terms of rotation matrices, rather than spherical harmonics, Legendre polynomials, Clebsch-Gordan coefficients, etc. as is the common practice. This was done because the correlation functions have a particularly simple form when presented in this way, but there is one drawback to this approach: although the rotation matrices are well known functions, they are not widely tabulated. For this reason, it seems appropriate to include a summary of the properties and values of rotation matrices which are relevant to the type of calculation outlined above. Throughout this paper, the notation of Rose [8], and of Brink and Satchler [19] has been adopted for the rotation operator. This operator is related to that of Wigner [20] and of Edmonds [21] by: $D^j_{m'm}$ (Rose) $= D^j_{-m'-m}$ (W&E).

A.1. GENERAL FORMULAE

$$D^j_{m'm}(\varphi, \theta, \Psi) = e^{-im'\varphi} d^j_{m'm}(\theta) e^{-im\Psi}$$

with

$$\begin{aligned} d^j_{m'm}(\theta) = {} & [(j-m')!(j+m')!(j-m)!(j+m)!]^{\frac{1}{2}} \\ & \times \sum_k [(-1)^k((j-m'-k)!(j+m-k)!(k+m'-m)!k!)^{-1} \\ & \times (\cos \tfrac{1}{2}\theta)^{2j+m-m'-2k}(-\sin \tfrac{1}{2}\theta)^{m'-m+2k}] \end{aligned}$$

and the recursion relation is

$$d^j_{m'm}(\theta) = \left[\frac{j+m'}{j+m}\right]^{\frac{1}{2}} d^{j-\frac{1}{2}}_{m'+\frac{1}{2}m+\frac{1}{2}} \cos \tfrac{1}{2}\theta + \left[\frac{j-m'}{j+m}\right]^{\frac{1}{2}} d^{j-\frac{1}{2}}_{m'-\frac{1}{2}m+\frac{1}{2}} \sin \tfrac{1}{2}\theta.$$

A.2. SYMMETRY RELATIONS

$$\begin{aligned} d^j_{m'm}(\theta) &= (-1)^{m'-m} d^j_{-m'-m}(\theta), \\ &= (-1)^{j-m'} d^j_{-m'm}(\theta+\pi), \\ &= (-1)^{j-m} d^j_{m'-m}(\theta+\pi), \\ &= (-1)^{m'-m} d^j_{mm'}(\theta), \\ &= d^j_{mm'}(-\theta), \end{aligned}$$

A.3. SPECIAL VALUES

$$d^j_{jm}(\theta) = [(2j)!/(j+m)!(j-m)!]^{\frac{1}{2}}(\cos\tfrac{1}{2}\theta)^{j+m}(-\sin\tfrac{1}{2}\theta)^{j-m},$$
$$d^j_{m0}(\theta) = (-1)^m[(j-|m|)!/(j+|m|)!]^{\frac{1}{2}}P_j^{|m|}(\cos\theta),$$
$$d^j_{00}(\theta) = P_j(\cos\theta),$$
$$D^j_{m0}(\varphi, \theta, \Psi) = (-1)^m[4\pi/(2j+1)]^{\frac{1}{2}}Y_j^{-m}(\theta, \varphi).$$

A.4. VALUES OF $d^j_{m\sigma}(\theta)$ FOR $j \leqq 4$, $\sigma = 0$ AND 1, AND $j-m$ EVEN

j	m	$\sigma = 0$	$\sigma = 1$
0	0	1	1
1	1	$-\sqrt{\frac{1}{2}}\sin\theta$	$\frac{1}{2}(1+\cos\theta)$
2	0	$\frac{1}{2}(3\cos^2\theta-1)$	$+\sqrt{\frac{3}{2}}\sin\theta\cos\theta$
2	2	$\frac{1}{2}\sqrt{\frac{3}{2}}\sin^2\theta$	$-\frac{1}{2}\sin\theta\,(1+\cos\theta)$
3	1	$-\frac{1}{4}\sqrt{3}\sin\theta(5\cos^2\theta-1)$	$\frac{1}{8}(1+\cos\theta)(15\cos^2\theta-10\cos\theta-1)$
3	3	$-\frac{1}{4}\sqrt{5}\sin^3\theta$	$\frac{1}{8}\sqrt{15}\sin\theta\,(1+\cos\theta)$
4	0	$\frac{1}{8}(35\cos^4\theta-30\cos^2\theta+3)$	$\frac{1}{4}\sqrt{5}\sin\theta\cos\theta\,(7\cos^2\theta-3)$
4	2	$\frac{1}{8}\sqrt{10}\sin^2\theta\,(7\cos^2\theta-1)$	$-\frac{1}{8}\sqrt{2}\sin\theta(1+\cos\theta)(14\cos^2\theta-7\cos\theta-1)$
4	4	$\frac{1}{16}\sqrt{70}\sin^4\theta$	$\frac{1}{8}(\sqrt{14}/8)\sin^3\theta\,(1+\cos\theta)$

A.5. ZEROES IN THE ROTATION MATRIX

Below are tabulated, to the nearest degree, angles at which the rotation matrices tabulated in Section D have the value zero.

j	m	σ	θ for $d(\theta) = 0$
0	0	0	none
1	1	0	0°, 180°
1	1	1	180°
2	0	0	55°, 125°
2	0	1	0°, 90°, 180°
2	2	0	0°, 180°
2	2	1	0°, 180°
3	1	0	0°, 63°, 117°, 180°
3	1	1	41°, 95°, 180°
3	3	0	0°, 180°
3	3	1	0°, 180°
4	0	0	30°, 70°, 110°, 150°
4	0	1	0°, 49°, 90°, 131°, 180°
4	2	0	0°, 68°, 112°, 180°
4	2	1	0°, 52°, 97°, 180°
4	4	0	0°, 180°
4	4	1	0°, 180°

Appendix B

THREE-DIMENSIONAL NUCLEAR DISINTEGRATION KINEMATICS

When a nucleus disintegrates by particle emission or fission following a nuclear reaction, the kinematical situation becomes considerably more complicated than in the case of a simple reaction. Consider a reaction of the type X(a, b)Y(c)Z, i.e., particle a bombards target X, yielding reaction particle b and excited nucleus Y, which subsequently disintegrates into breakup particle c and residual nucleus Z. Assume that the target position and beam direction are well defined and that particles b and c are detected in coincidence. The beam axis and the position of detector b will define the reaction plane. If detector c lies in this plane the kinematical situation is

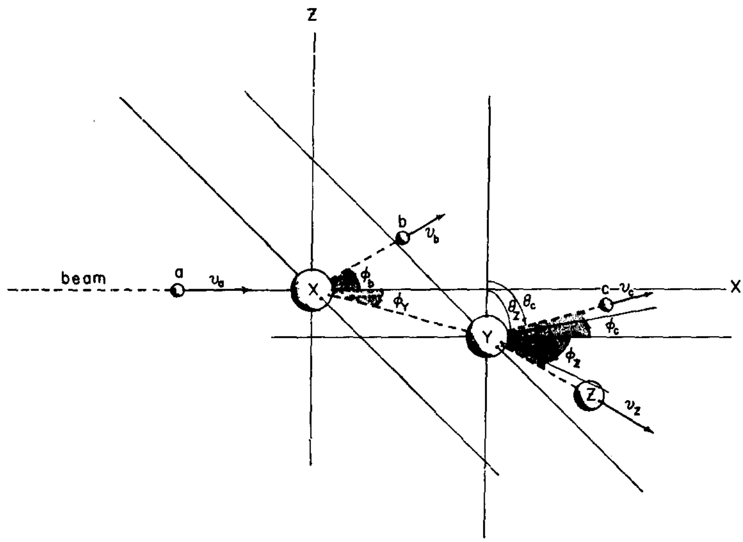

Fig. 3. Diagram illustrating the reaction X(a, b)Y(c) Z in the laboratory coordinate system. Velocities v_a, v_b, and v_y lie in the x, y-plane while v_c, and v_z do not. The directions of the latter must therefore be specified by the polar angles θ_c and θ_z, as well as the azimuthal angles ϕ_c and ϕ_z.

two-dimensional and some simplification is possible. An extensive treatment of this case has been given elsewhere [22]).

If, however, the particle detector c lies outside the reaction plane, three-dimensional kinematics must be employed. Fig. 3 shows this situation. In working out the kinematical relationships of this problem, three systems must be considered: the laboratory system (lab), the overall centre-of-mass system (CMS), and the centre-of-mass system of the recoiling excited nucleus Y (RCM). Angular coordinates θ and ϕ will be used to specify the polar and azimuthal coordinates in a given spherical polar coordinate system, lower-case letters indicating angles measured in the lab system and upper case those measured in the CMS or RCM system. The same convention will be applied to velocities v or V and solid angles $d\omega$ or $d\Omega$. The mass and laboratory energy of the particles will be indicated by m and E. Subscripts will refer to particles in the reaction. The Q values Q_r and Q_b will refer to the reaction and breakup Q values. The forms of the equations used here were adopted because of their simplicity and symmetry and their adaptability to FORTRAN computer programmes.

Since the derivation of two-dimensional reaction kinematical relationships has been given in a number of places [23, 24]) only the result will be given. First it is necessary to define the constant G:

$$G = (V_b/v_{CMS})^2 = \frac{m_Y}{m_a m_b E_a}[Q_r(m_Y+m_b)+E_a(m_Y+m_b-m_a)]. \tag{20}$$

Here v_{CMS} is the lab velocity of the centre-of-mass of the system, and V_b is the CMS velocity of particle b. If Q_r is negative it is possible that, for small values of E_a, G can be less than zero; this indicates that there is insufficient CMS energy to produce the reaction, i.e., E_a is less than the threshold energy. If G is less than 1, two particle groups may be observed at the same lab angle ϕ_b and these will have different laboratory energies E_b and centre- of mass angles Φ_b. At laboraty angles ϕ_b such that $\sin^2\phi_b > G$, no particle groups will be observed. In terms of G, the lab energy E_b and CMS angle Φ_b are

$$E_b = \frac{m_a m_b E_a}{(m_Y+m_b)^2}(\cos\phi_b \pm \sqrt{G-\sin^2\phi_b})^2, \tag{21}$$

$$\Phi_b = \phi_b + \operatorname{arctg}\left[\frac{\sin\phi_b}{\pm\sqrt{G-\sin^2\phi_b}}\right]. \tag{22}$$

The $\pm$ signs correspond to the double values mentioned above when G is less than 1; only the positive root should be used for values of G which are greater than 1. The solid angle of the detector must be corrected from the lab to the CMS. The ratio of the two solid angles is

$$\left(\frac{d\omega}{d\Omega}\right)_{CMS} = \frac{\sin^3\phi_b}{\sin^3\Phi_b}\left(1+\frac{\cos\Phi_b}{\sqrt{G}}\right). \tag{23}$$

The lab and CMS angles and lab energy of the recoil nucleus Y are related to the above relations by

$$E_Y = E_a - E_b + Q_r, \tag{24}$$

$$\phi_Y = \operatorname{arctg}[\sin\phi_b((m_Y E_Y/m_b E_b) - \sin^2\phi_b)^{-\frac{1}{2}}], \tag{25}$$

$$\Phi_Y = \pi - \Phi_b. \tag{26}$$

It should be noted that these equations will be double-valued whenever E_b and Φ_b are double-valued.

Thus far only the reaction-plane kinematics arising from the initial nuclear reaction have been considered. Now the relationships given above can be used to calculate the remaining unknown quantities θ_c, E_c, the RCM azimuthal angle Φ_c, and the solid angle correction $(d\omega/d\Omega)_c$ in the RCM system. It will be assumed that Q_b is known and that the RCM polar angle Θ_c is specified. It may seem strange that a centre-of-mass angle is to be specified and a lab angle calculated, but sect. 5 of this paper showed the simplifications in the correlation function which can be achieved at certain fixed RCM polar angles (see eq. (17) for example). Since the lab polar angle θ_c will be dependent on both the lab azimuthal angle ϕ_c and the RCM polar angle Θ_c, it may be necessary to change θ_c to the appropriate calculated value with every change in ϕ_c in order to maintain Θ_c at an essentially constant value.

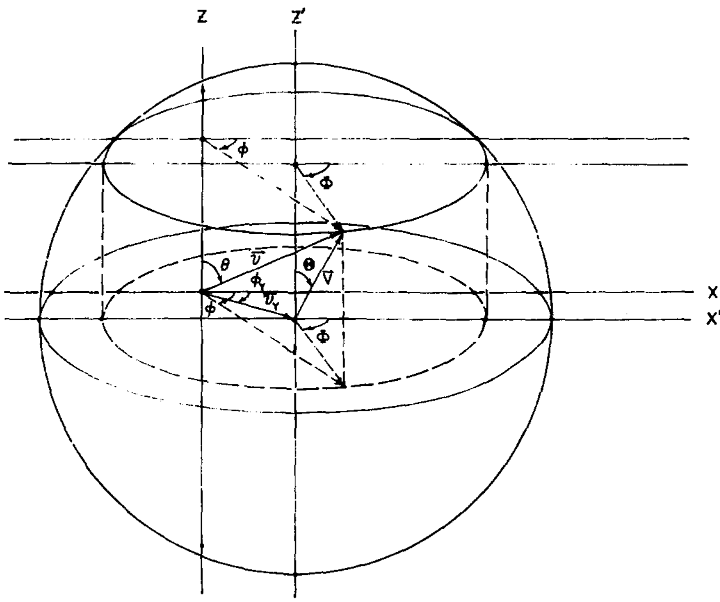

Fig. 4. Diagram illustrating the relation of the laboratory (lab) and recoil-centre-of-mass (RCM) coordinate systems. Particle RCM velocity V traces a sphere. The locus of velocities with RCM polar angle Θ constant is a circle on this sphere. The lab angles θ and ϕ depend on both RCM angles Θ and Φ.

Fig. 4 illustrates the vectorial relationships between $\boldsymbol{v}_Y$, the lab velocity of the RCM system, $\boldsymbol{v}_c$ the lab velocity of the breakup particle, and $\boldsymbol{V}_c$ the RCM velocity of the breakup particle. Since V_c is a constant in the RCM system, depending only on the breakup energy Q_b and the mass of the breakup particle m_c, V_c will trace out a sphere as shown. The locus of velocities V_c having Θ_c constant will trace a circle on this sphere. The X-axes shown correspond to the beam direction, while the Z-axes are the perpendiculars to the reaction plane; the unprimed axes are for the lab system and the primed axes for the RCM system. The projections of the three vectors on the three axes give the following relations:

$$v_Y \cos \phi_Y + V_c \sin \Theta_c \cos \Phi_c = v_c \sin \theta_c \cos \phi_c, \tag{27x}$$

$$v_Y \sin \phi_Y + V_c \sin \Theta_c \sin \Phi_c = v_c \sin \theta_c \sin \phi_c, \tag{27y}$$

$$V_c \cos \Theta_c = v_c \cos \theta_c. \tag{27z}$$

As in the reaction case, there is a constant which depends on the Q value of the reaction and plays an important part in the kinematical relationships. This constant will be called F; is is defined by

$$F = (V_c/v_Y)^2 = \frac{m_z}{m_Y m_c E_Y} [Q_b(m_z+m_c)+E_Y(m_z+m_c-m_Y)]$$
$$\approx (m_z Q_b/m_c E_Y), \quad \text{when} \quad Q_b \ll 931 \text{ MeV}. \tag{28}$$

From (27x) and (27y) the relation for Φ_c can be obtained:

$$\Phi_c = \phi_c + \text{arctg}[(F \sin^2 \Theta_c - \sin^2 (\phi_c - \phi_Y))^{-\frac{1}{2}} \sin(\phi_c - \phi_Y)]. \tag{29}$$

When $F \sin^2 \Theta_c$ is less than 1, Φ_c will be double valued, in analogy with (22).

The relation which gives the lab polar angle θ_c is obtained by eliminating the first term in (27x) and (27y) and then dividing through by (27z):

$$\theta = \text{arctg}[\text{tg}\, \Theta_c \sin(\Phi_c - \phi_Y)/\sin(\phi_c - \phi_Y)]. \tag{30}$$

The lab energy E_c of the breakup particle and the solid angle correction $(\mathrm{d}\omega/\mathrm{d}\Omega)_{\text{RCM}}$ could be derived from eqs. (27), but there is a simpler way of obtaining these quantities. Eqs. (21)-(26) were derived assuming a reaction of the form X(a, b)Y; now if this reaction is replaced by one of the form 0(Y, c)Z, where the 0 indicates that no target particle is involved, then the desired relationships can be obtained by simply changing the subscripts of the appropriate equations. However, one must also substitute F for G and Q_b for Q_r and note that the angle Ψ_c which is the laboratory angle between the "beam" direction and the "reaction-particle" direction, i.e., between the directions of motion of particle Y and particle c, is given by the spherical trigonometric relationship:

$$\cos \Psi_c = \sin \theta_c \cos(\phi_c - \phi_Y). \tag{31}$$

Replacing ϕ_b with this angle in (21-6) gives:

$$E_c = \frac{m_Y m_z E_Y}{(m_c+m_z)^2} (\cos \Psi_c \pm \sqrt{F+\cos^2\Psi_c - 1})^2, \tag{32}$$

$$\left(\frac{\mathrm{d}\omega}{\mathrm{d}\Omega}\right)_{\text{RCM}} = \frac{\sin^3 \phi_c}{\sin^3 \Phi_c} (1+F^{-\frac{1}{2}} \cos \Phi_c), \tag{33}$$

$$E_z = E_Y + Q_b - E_c, \tag{34}$$

$$\phi_z = \text{arctg}[((m_z E_z/m_c E_c) - \sin^2 \phi_c)^{-\frac{1}{2}} \sin \phi_c], \tag{35}$$

$$\Phi_z = \pi - \Phi_c. \tag{36}$$

As before, if F is less than 1 there will be two particle groups observed when $\sin^2 \Psi_c$ is less than F and no particles observed when $\sin^2 \Psi_c$ is greater than F. When there are two groups the $\pm$ sign in (29) and (32) should be used; when there is only one group only the + sign is meaningful.

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