On Market Games with Time, Location, and Free Disposal Constraints

Camelia Bejan[a], Juan Camilo Gómez*[,b]

[a]*Department of Economics, Rice University, 6100 Main Street, Houston, TX 77005*
[b]*Business Program, University of Washington Bothell*
18115 Campus Way NE, Bothell, WA 98011-8246

Abstract

We consider direct markets, pure exchange economies in which agents supply their time, when they are subject to location, time, and free disposal constraints. If agents' entire time endowment has to be devoted to the market (no free-disposal), such economies generate the entire class of TU-games, proving that every TU-game is a market game, and thus generalizing Shapley and Shubik's (1969) and Garratt and Qin's (2000) results. Markets in which free-disposal of time is possible (i.e., agents can be idle or devote part of their time to non-market activities) generate the class of monotonic TU-games.

Key words: TU market games, direct markets, equivalence between markets and games

1. Introduction

This paper belongs to the branch of literature initiated by Shapley and Shubik's seminal work relating economies and coalitional transferable utility (TU) games. Shapley and Shubik (1969) proved that a TU-game can arise from a pure-exchange economy (called a *market*) if and only if it is totally balanced. We prove here that, if markets are subject to time, location, and free-disposal constraints, a game does not need to be totally balanced to be a market game. In fact, every TU game is a market game. Our approach does not require the existence of a large number of players (see Wooders (1994)) or public goods (see Meseguer-Artola, Wooders, and Martinez-Legaz (2003)).

Shapley and Shubik (1969) propose procedures to associate each game with a market and each market with a game. Given a TU-game v, they construct a particular exchange economy called the *direct market* of v. In such direct market, agents are the players of v and they buy and sell their productive time. Utilities are measured in units of money, and agents can exchange commodities and transfer money in any amount. Utility functions are identical for all agents and

*Corresponding author

maximize aggregate production by coalitions given a certain vector of productive times. On the other hand, a market game is a cooperative game generated from an economy. It sets the worth of every coalition equal to the maximum utility that coalition can achieve by using the endowments of its members. Shapley and Shubik (1969) show that every totally balanced game is the game generated by its direct market, and that every game generated from a market must be totally balanced.

Garratt and Qin (2000) pointed out that since timing considerations have an impact on the feasible allocations in a market, they also affect the class of games that can be generated from markets. As in Shapley and Shubik (1969), they consider direct markets in which time can be divided among various productive activities. However, they assume that the production process requires the physical presence of agents and thus, agents cannot be in two places at once. Besides, all productive activities must take place during a predetermined time interval. They call such economies *time-constrained* and prove that there is a one-to-one and onto mapping between these and the family of super-additive TU-games, a strict superset of the class of totally balanced games.

In this paper we show that by taking into account not only time, but also location (or space) constraints in a market, every TU-game can be represented as a market game. Starting from Garratt and Qin's (2000) framework, we consider the case in which all productive activities require the *exclusive* use of a given facility. For example, imagine a conference room which can be booked for various talks, but cannot be used for two (or more) talks simultaneously. If we assume, as in the previous literature, that agents' time endowments must be entirely devoted to the market (i.e., no free-disposal of time) then every TU-game can arise from such a market. Relaxing this assumption by considering markets in which some agents may remain idle for a fraction of their time leads to monotonic TU-games.

The paper is organized as follows. In Section 2 we introduce some notation and basic definitions. Economies with time and location constraints are analyzed in Section 3, where it is proved that every TU-game is a market game arising from an economy without free disposal, and that markets with free-disposal generate monotone games. Section 4 concludes.

2. Definitions and Notation

Let N be a finite set of n players and $\mathcal{N}$ the family of all non-empty subsets of N. Let Δ_N the unit simplex in $\mathbb{R}^N$ and $\Delta_{\mathcal{N}}$ the unit simplex in $\mathbb{R}^{\mathcal{N}}$. For every $i \in N$, let $e_i \in \Delta_N$ be the vertex corresponding to i. For every $S \subseteq N$, let $\mathbf{1}_S \in \{0,1\}^N$ denote the indicator function of S and let $e_S \in \Delta_{\mathcal{N}}$ be the vertex corresponding to S.

A *TU-game* (or simply a *game*) on N is a mapping $v : 2^N \to \mathbb{R}_+$ such that $v(\emptyset) = 0$. Let Γ^N denote the family of games on N. For every $S \subseteq N$, $v(S)$ is called the *worth of coalition* S. The restriction of a game v to $S \subseteq N$, is the game $v_{|S}$ on S such that $v_{|S}(T) := v(T)$ for all $T \subseteq S$. Given $v \in \Gamma^N$, a possible outcome is represented by a *payoff vector* $u \in \mathbb{R}^N$ that assigns to every $i \in N$ a payoff u_i.

A game $v \in \Gamma^N$ is called *superadditive* if $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \in \mathcal{N}$ such that $S \cap T = \emptyset$. It is called *monotonic* if $v(T) \leq v(S)$ for all $\emptyset \neq T \subseteq S \subseteq N$. A collection of coalitions $\mathcal{B} \subseteq \mathcal{N}$ is called *balanced* if there exist positive numbers $(\lambda_S)_{S \in \mathcal{B}}$ such that for every $i \in N$, $\sum_{S \in \mathcal{N}_i} \lambda_S = 1$. The numbers λ_S are called *balancing weights*. A game $v \in \Gamma^N$ is called *balanced* if $\sum_{S \in \mathcal{B}} \lambda_S v(S) \leq v(N)$ for every balanced family $\mathcal{B}$ with balancing weights $(\lambda_S)_{S \in \mathcal{B}}$. A game v is called *totally balanced* if $v_{|S}$ is balanced for every $S \subseteq N$.

3. Time- and location-constrained market games

The procedure we follow to characterize games generated by markets is used by Shapley and Shubik (1969). We start with an arbitrary game and use it to construct a direct market. Then we use the direct market to generate a TU-game. Finally, we show that the initial game and the market generated game are identical. The construction of our market is similar to Garratt and Qin's (2000), with an added location constraint on the set of feasible time arrangements.

3.1. No free-disposal and arbitrary TU-games

Let $v \in \Gamma^N$. We associate to v a market (or pure exchange economy) $\mathcal{E}(v)$ as follows. The set of agents in the market is N and the commodity space is $[0,1]^N$. Commodities are interpreted as agent-specific time. Each agent $i \in N$ has an endowment e_i (thus, each agent is endowed with one unit of her own time) and a (common) utility function $u : [0,1]^N \to \mathbb{R}_+$ defined for every $x \in [0,1]^N$ by:

$$u(x) = \max_{\lambda:\mathcal{N}\to\mathbb{R}_+} \sum_{S \in \mathcal{N}} \lambda_S v(S), \text{ subject to} \tag{1}$$

$$\sum_{S \in \mathcal{N}} \lambda_S \mathbf{1}_S = x, \tag{i}$$

$$\sum_{S \in \mathcal{N}} \lambda_S \leq 1. \tag{ii}$$

For every $x \in [0,1]^N$, let the set of feasible *schedules* be denoted by $\mathcal{F}(x) := \{\lambda : \mathcal{N} \to \mathbb{R}_+ \mid \lambda \text{ satisfies (i) and (ii)}\}$ and let $A(x) \subseteq \mathcal{F}(x)$ denote the set of solutions of (1).

Remark 3.1 *Note that the utility function is well-defined for every $x \in [0,1]^N$ and quasi-concave. Indeed, the function $\lambda \mapsto \sum_{S \in \mathcal{N}} \lambda_S v(S)$ is linear, hence continuous, and the feasible set $\mathcal{F}(x)$ is clearly compact. To see that $\mathcal{F}(x)$ is non-empty, let $x \in [0,1]^N$ and define, recursively, $m_0 = 0$,*

$$m_k = \min\{x_i - \sum_{j=0}^{k-1} m_j \mid i \in N,\ x_i > \sum_{j=0}^{k-1} m_j\},$$

$$S_k = \{i \in N \mid x_i \leq \sum_{j=1}^{k} m_j\},$$

and $\lambda_{S_k} = m_k$ *for every* $1 \leq k \leq |\{x_i \mid i \in N\}|$*. Let* $\lambda_S = 0$ *for every other* $S \in \mathcal{N}$*. By construction* λ *satisfies* (i) *and* $\sum_{S\in\mathcal{N}} \lambda_S = \max_{i\in N} x_i \leq 1$*, so* $\lambda \in \mathcal{F}(x)$ *and the utility is well-defined. For quasi-concavity, consider* $x, \bar{x} \in [0,1]^N$ *and* $\alpha \in [0,1]$*. Let* $\lambda \in \mathcal{F}(x)$ *and* $\bar{\lambda} \in \mathcal{F}(\bar{x})$*. Then* $\alpha\lambda + (1-\alpha)\bar{\lambda} \in \mathcal{F}(\alpha x + (1-\alpha)\bar{x})$ *and thus* $u(\alpha x + (1-\alpha)\bar{x}) \geq \alpha u(x) + (1-\alpha)u(\bar{x})$*.*

To understand the definition of the utility function it is helpful to think of the agents as being managers in charge of deciding how to schedule operating coalitions at a given facility during a limited time interval. It is assumed that every coalition needs to use the facility to perform its productive activity, and two coalitions cannot use the facility at the same time. We call this restriction the *location constraint*. The facility is also open only for a limited time and thus all productive activities have to be scheduled during that time interval. We call this the *time constraint*. However, the time that each member of a coalition supplies is perfectly divisible, and each individual can divide her time among various operating coalitions. It is also assumed that, given the right to use the space of the facility, each coalition has a constant returns to scale technology. That is, a coalition S that uses the facility for a fraction λ_S of the time produces $\lambda_S v(S)$ units of output. A vector $x \in [0,1]^N$ is interpreted as unscheduled time, and thus any solution of (1) is an optimal schedule under the given location and time constraints. The weight λ_S specifies the amount of time during which coalition S is active. The first constraint is the requirement that each agent i is scheduled for *exactly* x_i units of her available time (thus, time is not disposable), while the second constraint controls the time during which the location is available. Compared to Garratt and Qin's (2000) formulation, the definition of the utility function allows for only one coalition to be active at any moment, capturing the idea of space limitation.

The following lemma shows that, given time and location constraints, the only feasible way to allocate the full time of the members of coalition S is to have the entire coalition active.

Lemma 3.2 *For every* $v \in \Gamma^N$ *and every* $S \in \mathcal{N}$*,* $u(\mathbf{1}_S) = v(S)$*.*

Proof. Let $\lambda \in \mathcal{F}(\mathbf{1}_S)$. According to (i), for every $i \notin S$, $\sum_{T\ni i} \lambda_T = 0$ and thus $\lambda_T = 0$ for every $T \nsubseteq S$. On the other hand, for every $i \in S$, $\sum_{T\ni i} \lambda_T = 1$. Choosing an arbitrary $i \in S$ and $T_0 \subset S$ such that $i \notin T_0$ we obtain, using (ii), that

$$1 \geq \sum_{T\subseteq S} \lambda_T \geq \sum_{T\ni i} \lambda_T + \lambda_{T_0} \geq 1 + \lambda_{T_0}$$

and thus $\lambda_{T_0} = 0$. Hence $\lambda_T = 0$ for every $T \neq S$ and $\lambda = e_S$. We then conclude that $u(\mathbf{1}_S) = v(S)$. ■

Remark 3.3 *In the light of Lemma 3.2, the problem of a manager in charge of allocating the time vector* $\mathbf{1}_S$ *is trivial, since the feasible set is a singleton. This* *is not typically the case for time-vectors $x \neq \mathbf{1}_S$, $S \in \mathcal{N}$. Consider for example $N = \{1,2\}$ and $x = (\frac{1}{2}, \frac{1}{2})$. The feasible set of a manager in charge of allocating x is*

$$\mathcal{F}\left(\frac{1}{2}, \frac{1}{2}\right) = \left\{(\lambda_1, \lambda_2, \lambda_{12}) = \left(\frac{1}{2} - \alpha, \frac{1}{2} - \alpha, \alpha\right) \mid \alpha \in \left[0, \frac{1}{2}\right]\right\}.$$

Next, we use the location- and time-constrained direct market to generate a TU-game. We associate to the economy $\mathcal{E}(v)$ the game $w \in \Gamma^N$ defined as follows. For every $S \in \mathcal{N}$,

$$w(S) = \max_{(x^i)_i \in X^S} \sum_{i \in S} u(x^i), \tag{2}$$

where

$$X^S = \left\{ (x^i)_{i \in S} \;\middle|\; \begin{array}{l} \forall i \in S,\ x^i \in [0,1]^N, \\ \sum_{i \in S} x^i = \mathbf{1}_S, \\ \forall i \in S\ \exists \lambda^i \in A(x^i) \text{ s.t. } \sum_{i \in S} \lambda^i \in \mathcal{F}(\mathbf{1}_S). \end{array} \right\}$$

To interpret the characteristic function w, agents in economy $\mathcal{E}(v)$ are seen as managers who schedule the operating coalitions optimally. Thus, to compute $w(S)$, the time endowment of coalition S is first divided among its members. Each manager $i \in S$ is allocated the right to manage a time-vector $x^i \in [0,1]^N$, such that $\sum_{i \in S} x^i = \mathbf{1}_S$. The optimal schedules chosen by managers ($\lambda^i \in A(x^i)$) can be pooled together to allocate the entire time endowment of coalition S if and only if they are compatible with the time and location constraints, that is, if $\sum_{i \in S} \lambda^i \in \mathcal{F}(\mathbf{1}_S)$. Thus $w(S)$ is the maximum surplus generated by distributions of the total time endowment for which managers' solutions to (1) are compatible.

The following theorem shows that every game $v \in \Gamma^N$ can be generated from its time- and location-constrained direct market $\mathcal{E}(v)$. Thus a game does not need to be totally balanced, or even super-additive to be a market game. *Every* TU-game is a market game.

Theorem 3.4 *For every $v \in \Gamma^N$ and every $S \subseteq N$, $w(S) = v(S)$.*

Proof. Fix $S \in \mathcal{N}$. Let $(x^i)_{i \in S} \in X^S$ be an optimal solution to (2) and let $(\lambda^i)_{i \in S}$ be a family of optimal compatible schedules, i.e., $\lambda^i \in A(x^i)$ for every $i \in S$ and $\sum_{i \in S} \lambda^i \in \mathcal{F}(\mathbf{1}_S)$. Then, according to Lemma 3.2,

$$v(S) = u(\mathbf{1}_S) = \sum_{T \subseteq N} \left(\sum_{i \in S} \lambda^i_T\right) v(T) = \sum_{i \in S} \left(\sum_{T \subseteq N} \lambda^i_T v(T)\right) = \sum_{i \in S} u(x^i) = w(S).$$

Therefore every TU-game coincides with the game generated by its direct market. ■

3.2. Free disposal and monotonic games

We now consider markets with free disposal, that is, markets in which agents can remain idle or, equivalently, devote a fraction of their time to non-market activities. In this case we show that any monotonic game can be obtained as a market game with free disposal. As before, we define a direct market for every TU-game and associate a TU-game to every direct market.

In this case, the direct market associated with a game $v \in \Gamma^N$ has the same set of agents and commodity space as before, but the (common) utility function of the agents is $u' : [0,1]^N \longrightarrow \mathbb{R}_+$ defined by:

$$u'(x) = \max_{\lambda:\mathcal{N}\to\mathbb{R}_+} \sum_{S\subseteq N} \lambda_S v(S), \text{ subject to} \tag{3}$$

$$\sum_{S\subseteq N} \lambda_S \mathbf{1}_S \leq x, \tag{i'}$$

$$\sum_{S\subseteq N} \lambda_S \leq 1. \tag{ii'}$$

As opposed to the definition in the previous section, constraint (i') may not be binding so, this direct market allows for the free disposal of agents' time. Let $\mathcal{F}'(x)$ denote the feasible set of schedules for the problem (3) and let $A'(x)$ be the set of its solutions. In a similar way to Remark 3.1, u' is well defined and quasi-concave.

The following lemma shows that an optimal way to allocate the time of the members of coalition S is to have the most productive sub-coalition of S active.

Lemma 3.5 *For every $v \in \Gamma^N$ and every $S \in \mathcal{N}$,*

$$u'(\mathbf{1}_S) = \max\{v(T) \mid T \subseteq S\}.$$

Proof. Let $S \in \mathcal{N}$. According to (i'), for every $i \notin S$, $\sum_{Q\ni i} \lambda_Q = 0$ and thus $\lambda_Q = 0$ for every $Q \not\subseteq S$. Hence $u'(\mathbf{1}_S) = \sum_{T\subseteq S} \lambda^*_T v(T)$ for some $\lambda^* : \mathcal{N} \to \mathbb{R}_+$ satisfying (i') and (ii'), which implies that $u'(\mathbf{1}_S) \leq \max\{v(T) \mid T \subseteq S\}$. On the other hand, for every $T \subseteq S$, $e_T \in \mathcal{F}'(\mathbf{1}_S)$, and thus $u'(\mathbf{1}_S) \geq \max\{v(T) \mid T \subseteq S\}$. ■

We associate to this direct market the following game $w' \in \Gamma^N$. Define for every $S \in \mathcal{N}$:

$$w'(S) = \max_{(x^i)_{i\in\bar{X}^S}} \sum_{i\in S} u'(x^i), \tag{4}$$

where

$$\bar{X}^S = \left\{ (x^i)_{i\in S} \;\middle|\; \begin{array}{l} x^i \in [0,1]^N, \forall i \in S, \\ \sum_{i\in S} x^i \leq \mathbf{1}_S, \\ \forall i \in S\ \exists \lambda^i \in A'(x^i) \text{ s.t. } \sum_{i\in S} \lambda^i \in \mathcal{F}'(\mathbf{1}_S). \end{array} \right\}$$

Proposition 3.6 *For every* $v \in \Gamma^N$ *and every* $S \in \mathcal{N}$*,* $w'(S) = u'(\mathbf{1}_S) = \max\{v(T) \mid T \subseteq S\}$*.*

Proof. Fix $S \in \mathcal{N}$. Let $\hat{T} \in \arg\max\{v(T) \mid T \subseteq S\}$ and fix a player $j \in S$. Define $\hat{x}^j = \mathbf{1}_{\hat{T}}$ and $\hat{x}^i = 0$ for every $i \in S \setminus \{j\}$. Additionally, define $\hat{\lambda}^j_{\hat{T}} = 1$ and $\hat{\lambda}^i_T = 0$ for every other pair $(i, T) \in S \times \mathcal{N}$. Then, $\hat{x}^i \in A'(\hat{x}^i)$ for all $i \in S$, $\sum_{i \in S} \hat{\lambda}^i \in \mathcal{F}'(\mathbf{1}_S)$, and therefore $\hat{x} \in \bar{X}^S$. Hence, $w'(S) \geq \sum_{i \in S} u'(\hat{x}^i) = u'(\mathbf{1}_S)$.

Conversely, let $(\bar{x}^i)_{i \in S}$ be an optimal solution to (4) and let $\bar{\lambda} \in \prod_{i=1}^{|S|} A'(\bar{x}^i)$ such that $\sum_{i \in S} \bar{\lambda}^i \in \mathcal{F}'(\mathbf{1}_S)$. Then, according to Lemma 3.5,

$$
\begin{aligned}
w'(S) &= \sum_{i \in S} u'(\bar{x}^i) = \sum_{i \in S} \left(\sum_{T \subseteq N} \bar{\lambda}^i_T v(T) \right) \\
&= \sum_{T \subseteq N} \left(\sum_{i \in S} \bar{\lambda}^i_T \right) v(T) \leq u'(\mathbf{1}_S)
\end{aligned}
$$

Thus, $w'(S) = u'(\mathbf{1}_S)$ as we wanted. ■

The following theorem is a direct consequence of Proposition 3.6 and the definition of a monotonic game.

Theorem 3.7 *Every monotonic game* $v \in \Gamma^N$ *is a location- and time-constrained market with free disposal.*

4. Concluding remarks

We proved that by considering economies with space restrictions in addition to time constraints, every TU-game can be generated as a market game arising from an economy with no free disposal, and every monotonic game arises as the game generated by a free-disposal market. The paper points out to a relationship between the feasible set of a direct market and the class of games generated by that market. Markets with smaller feasible set ceteris paribus, generate a larger class of games. As Shapley and Shubik (1969) proved, markets with no free-disposal generate the class of totally balanced games. No free-disposal markets with time constraints generate the family of super-additive games (a strict superset of totally balanced games), while no free-disposal markets with both time and location constraints generate the entire family of TU-games. Free disposal markets with time and location constraints generate the class of monotonic TU-games.

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