Using the Core to Predict Coalition Formation

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Abstract

The main objective of this work is to use the defining principles of the core solution concept to determine not only payoffs but coalition formation. Given a cooperative transferable utility (TU) game, we propose two non-cooperative procedures that deliver a natural and non-empty core extension, the aspiration core (Cross 1967, Bennett 1983), together with the supporting coalitions it implies. As expected, if the cooperative game is balanced, the grand coalition forms. However, if the core is empty, other coalitions arise. Following the aspiration literature, not only partitions but also overlapping coalition configurations are allowed. Our procedures interpret this fact in different ways. The first game allows players to participate simultaneously in more than one coalition, while the second assigns probabilities to the formation of potentially overlapping coalitions. We use the strong Nash and subgame perfect Nash equilibrium concepts.

Keywords: non-empty core extension, strategic coalition formation, aspiration core

JEL Classification Numbers: C71, C72.

1 Introduction

The two main issues in cooperative game theory are payoff assignment and coalition formation. It is then remarkable that the core, by far the most utilized cooperative solution concept, essentially only addresses the first question. Which coalitions arise is a non-issue for balanced games: it can be assumed that the grand coalition always forms, since it generates the highest total surplus for the players of the game. However, if balancedness (or superadditivity) is not satisfied, it is desirable to endogenously determine not only payoffs but also coalitional outcomes.This paper adapts ideas from two non-cooperative core implementation procedures, Kalai, Postlewaite, and Roberts (1979) and Pérez-Castrillo (1994), to settings in which the grand coalition is not assumed to form. Our games deliver, as equilibrium outcomes, the aspiration core[1] vectors (Cross 1967, Bennett 1983) and their supporting coalitions. As expected if the game is balanced, the grand coalition forms and equilibrium payoffs coincide with the core vectors.

Aspirations can be interpreted as price vectors resulting from demands placed by players for participating in any coalition. An aspiration cannot be improved upon by any coalition but, compared to a core vector, has a much weaker feasibility requirement: each player's payoff must be low enough so at least one coalition is able to support it. The aspiration core is defined as the set of aspirations of minimal sum and it has been shown to be a non-empty core extension (Bennett 1983). In games with an empty core, the aspiration core payoffs are not feasible for the grand coalition. This opens up the possibility for other coalitions to form.

Coalitions supporting a given aspiration are not necessarily disjoint. The literature on coalition formation has generally discarded this type of outcome as lacking predicting power. Greenberg (1994), for example, noted: "Clearly,

[1]The aspiration core is also known as the balanced aspiration set.

the set of coalitions that support an aspiration is not a partition. Thus, despite its appeal, an aspiration fails to predict which coalition will form, and moreover, it ignores the possibility that players who are left out will decide to lower their reservation price." Both of our procedures deal with this criticism, albeit in different ways.

In our first game aspiration core payoffs are attained via fractional coalition participation. While belonging to a single coalition is an appropriate assumption to model some situations (one person cannot belong to the Democratic and Republican parties simultaneously), there are other instances in which it is reasonable for agents to divide their time, or other resource, among several coalitions. For example, people might work for more than one firm, invest in multiple ventures, or belong to a number of clubs. Such situations are not covered by solution concepts specifically designed to deal with partitions. Even Bennett (1983) proposed the eventual formation of a partition of coalitions. We depart from the traditional approach to coalition formation by allowing a single player to participate in multiple coalitions.

Our first game starts by players simultaneously announcing a payoff for themselves and a distribution of their time across coalitions. For example, a player might divide her available time among two or more non-disjoint coalitions. A coalition forms when all of its members choose to spend a positive fraction of their time in it, and their demands are less than or equal to its worth. The coalition is active for a period of time equal to the minimum participation by one of its members. A player who participates in a coalition that operates a fraction λ of time gets a fraction λ of his demand. It is assumed that time not spent in an active coalition is spent alone. Along the lines of an important branch of core implementation literature, (see Kalai, Postlewaite, and Roberts (1979), Hart and Kurz (1983), Borm and Tijs (1992) and Young (1998)), our first procedure focuses on strong Nash equilibria. We then show that equilibrium strategies in our game correspond to aspiration core payoffs and, conversely, that any aspiration core vector is an equilibrium outcome of our procedure. In equilibrium, players organize into efficient, overlapping coalition structures.

Our second game illustrates a probabilistic approach to the overlapping coalition issue. Instead of allowing for non-disjoint coalitions to appear, they are assigned a positive probability of actually forming. Ex-post, a single coalition forms. For example, if we consider the three-player simple majority game[2] our game's equilibrium outcome is that coalitions $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$ will each form with probability $\frac{1}{3}$.

Following a different current of non-cooperative support literature (see Pérez-Castrillo (1994), Morelli and Montero (2003), and more recently (Keiding and Pankratova 2010)) the second game is played by agents called principals. A first principal is the owner of a set of inputs, the players of the original cooperative game. She sells them to potential entrepreneurs in the market, who will set up firms (i.e., create "coalitions" of inputs) and operate them to produce output. The game has two stages. In the first stage, the interested entrepreneurs submit bids to the owner. A bid represents the *total* amount an entrepreneur is willing to pay to get the inputs. The entrepreneur with the highest bid wins and enters stage 2, while the others leave the game. In

[2]In this three-player game coalitions of two and three players have a worth equal to one and the remaining coalitions are worth zero.

the second stage the owner puts a price on each input so that the sum of the prices equals the winning bid, while the entrepreneur selects the set of inputs that she wants to buy. The transaction takes place, with the entrepreneur paying for the inputs she wants at the prices set by the owner. Once more, subgame perfect Nash equilibria (SPNE) of the game generate aspiration core vectors.

Our games are related to the proposal-making bargaining game introduced by Selten (1981) and extended by Bennett (1997) which, for a very particular class of games,[3] supports aspirations (semi-stable payoff vectors in Selten's (1981) terminology) as outcomes of stationary subgame perfect Nash equilibria. With additional restrictions, the outcome is refined to the set of partnered aspirations (or stable payoff vectors in Selten's (1981) terminology). However, our approach applies to a much more general class of games, generates efficient outcomes, and coincides with the core in balanced games.

The paper is organized as follows. Section 2 introduces notation and basic definitions, Sections 3 and 4 describe the non-cooperative games generating the aspiration core vectors and coalitions, and Section 5 concludes.

2 Definitions and Notation

Let $N = \{1, ..., n\}$ be a finite set of players, where $n \in \mathbb{N} \setminus \{0\}$. Let $\mathcal{N}$ be the collection of all non-empty subsets of N and, for every $i \in N$, let $\mathcal{N}_i = \{S \in \mathcal{N} \mid S \ni i\}$. Let Δ_N be the unit simplex in $\mathbb{R}^n$, $\Delta_{\mathcal{N}}$ the unit

[3] Selten (1981) requires that the complement of any productive coalition must have zero worth.

simplex in $\mathbb{R}^{2^n-1}$ and $\Delta_{\mathcal{N}_i} = \{\lambda \in \Delta_{\mathcal{N}} \mid \lambda_S = 0 \text{ if } i \notin S\}$. For any $S \in \mathcal{N}$, let $\mathbf{e}_S \in \Delta_{\mathcal{N}}$ be the vertex of $\Delta_{\mathcal{N}}$ corresponding to coalition S. A *TU-game* (or simply a *game*) with a finite set of players N is a mapping $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. For any $S \subseteq N$, $v(S)$ is called the *worth of coalition* S. The *zero-normalization* of v is a TU-game, v_0, with the same set of players N, such that for every $S \subseteq N$, $v_0(S) = v(S) - \sum_{i \in S} v(\{i\})$.

A possible outcome of the game v is represented by a *payoff vector* $x \in \mathbb{R}^n$ that assigns to every $i \in N$ a payoff x_i. Given $x \in \mathbb{R}^n$ and $S \subseteq N$, let $x(S) := \sum_{i \in S} x_i$, with the agreement that $x(\emptyset) = 0$. A payoff vector $x \in \mathbb{R}^n$ is *feasible* for coalition S if $x(S) \leq v(S)$. It is *aspiration feasible* if for every $i \in N$, there exists $S \subseteq N$ with $i \in S$ such that x is feasible for S. We say that a coalition S is able to *improve upon* the outcome $x \in \mathbb{R}^n$ if $x(S) < v(S)$. A vector $x \in \mathbb{R}^n$ is called *stable* if it cannot be improved upon by any coalition. The *core* of a game v, denoted $C(v)$, is the set of stable outcomes that are feasible for N i.e.,

$$C(v) := \{x \in \mathbb{R}^n \mid x(S) \geq v(S) \; \forall S \subseteq N, \; x(N) = v(N)\}.$$

A stable payoff vector $x \in \mathbb{R}^n$ that is aspiration feasible is called an *aspiration*. We denote by $Asp(v)$ the set of aspirations of game v. It is known that for any TU-game v, $Asp(v)$ is a non-empty, compact and connected set (Bennett and Zame 1988). The *generating collection* of an aspiration x is the family of coalitions S that can attain x, i.e.,

$$\mathcal{GC}(x) := \{S \in \mathcal{N} \mid x(S) = v(S)\}.$$

A collection of coalitions $\mathcal{B} \subseteq 2^N$ is called *balanced* (respectively *weakly balanced*) if every $S \in \mathcal{B}$ is associated with a positive (resp. non-negative) number λ_S such that for every $i \in N$, $\sum_{S \in \mathcal{B}, S \ni i} \lambda_S = 1$. The numbers λ_S are called *balancing weights*. It is customary to interpret the balancing weight λ_S as the fraction of resources each player (in S) devotes to coalition S, or as the fraction of time coalition S is active (see, for example, (Kannai 1992)). Thus, a pair $(\mathcal{B}, \lambda)$ consisting of a (weakly) balanced family $\mathcal{B}$ and balancing weights $\lambda := (\lambda_S)_{S \in \mathcal{B}}$ can be interpreted as a feasible *overlapping coalition structure*, that is, a family of coalitions that can co-exist if players can divide their resources/time. The *total surplus* generated by an overlapping coalition structure $(\mathcal{B}, \lambda)$ is $b(\mathcal{B}, \lambda) := \sum_{S \in \mathcal{B}} \lambda_S v(S)$. An overlapping coalition structure $(\mathcal{B}, \lambda)$ is called *stable* if there exists a stable $x \in \mathbb{R}^n$ such that $x(N) \leq b(\mathcal{B}, \lambda)$.

For every TU-game v we define $\bar{b}(v)$ as the maximum total surplus generated by an overlapping coalition structure:

$$\bar{b}(v) := \max\{b(\mathcal{B}, \lambda) \mid \mathcal{B} \text{ is balanced w.r.t. weights } \{\lambda_S\}\}. \tag{1}$$

A coalition structure $(\mathcal{B}, \lambda)$ is *efficient* if $b(\mathcal{B}, \lambda) = \bar{b}(v)$. It is known that $\bar{b}(v)$ is finite and $\bar{b}(v) = \min\{x(N) \mid x(S) \geq v(S) \; \forall S \subseteq N\}$, (Bennett 1983). This implies that an overlapping coalition structure is efficient if and only if it is stable. Moreover, the core of a game v is non-empty if and only if $\bar{b}(v) \leq v(N)$ (Bondareva 1963, Shapley 1967). Such games are known as *balanced* games.

The *aspiration core* of a game v is given by

$$AC(v) := \{x \in Asp(v) \mid \mathcal{GC}(x) \text{ is weakly balanced }\}.$$

The aspiration core is non-empty for every TU-game v and it coincides with the core whenever the latter is non-empty. Two alternative but equivalent definitions of the aspiration core are $AC(v) = \{x \in Asp(v) \mid x(N) = \bar{b}(v)\}$ and $AC(v) = \arg\min\{x(N) \mid x \in \mathbb{R}^n,\ x(S) \geq v(S)\ \forall S \subseteq N\}$ (See Bennett (1983)).

If $x \in AC(v)$ and $\mathcal{B} \subseteq \mathcal{GC}(x)$ is a weakly balanced family with associated weights λ, then $b(\mathcal{B}, \lambda) = x(N) = \bar{b}(v)$ and thus $(\mathcal{B}, \lambda)$ is efficient. Reciprocally, every efficient overlapping coalition structure $(\mathcal{B}, \lambda)$ is stable, which implies that there exists $x \in AC(v)$ such that $x(N) = \bar{b}(v) = b(\mathcal{B}, \lambda)$ and thus $\mathcal{B} \subseteq \mathcal{GC}(x)$. Hence, the only efficient (and stable) overlapping coalition structures are weakly balanced subsets of the generating collections of aspiration core allocations.

3 The Time Allocation Game

Fix a TU-game v with a finite set of players N. Given the covariance properties of the aspiration core, there is no loss of generality in assuming that v is zero-normalized and thus, $v(\{i\}) = 0$ for all $i \in N$.[4] We associate with v a non-cooperative game $\Gamma(v)$, defined as follows. The player set of $\Gamma(v)$ is N. Each player $i \in N$ is endowed with one unit of a resource (for example, time)

[4] The aspiration core is covariant with respect to affine transformations of the game, that is $AC(\alpha v + \beta) = \alpha AC(v) + \beta$ for every game v and every $\alpha \in \mathbb{R}_+, \beta \in \mathbb{R}^n$.

and has to decide how to allocate it among different coalitions. In other words, player i chooses a resource allocation vector $\lambda_i \in \Delta_{\mathcal{N}_i}$. Additionally, player i chooses a payoff for himself, $x_i \in \mathbb{R}$. Thus, player i's strategy space is $\mathcal{S}_i = \Delta_{\mathcal{N}_i} \times \mathbb{R}$. Let $\mathcal{S} = \prod_{i\in N} \mathcal{S}_i$ be the *strategy space* of the game.

A coalition is productive only as long as all its members want to invest resources in it and their payoffs are feasible for S. More formally, given a strategy profile $s = (\lambda, x) \in \mathcal{S}$, a coalition $S \in \mathcal{N}$ generates a total surplus for its members of $\lambda_0(S)v(S)$, where $\lambda_0(S) = \min_{i\in S} \lambda_i(S)$ if $x(S) \leq v(S)$ and $\lambda_0(S) = 0$ otherwise. The payoff to i at $s \in \mathcal{S}$ is $p_i(s) := \alpha_i(s)x_i$, where $\alpha_i(s) := \sum_{S\ni i} \lambda_0(S)$ is the fraction of time spent productively. Notice that, by definition, $\alpha_i(s) \in [0, 1]$ and thus $p_i(s) \leq x_i$, for all $i \in N$ and $s \in \mathcal{S}$. We think of $1 - \alpha_i(s)$ as the amount of time agent i is idle or equivalently, given that v is zero normalized, time dedicated to coalition $\{i\}$. Accordingly, we define the *set of formed coalitions* as

$$\mathcal{F}(s) := \{S \in \mathcal{N} \mid \lambda_0(S) > 0\} \cup \Big\{ \{i\} \mid \alpha_i(s) < 1\Big\}.$$

For any strategic-form game with a finite player set N and any strategy s we say that a coalition S *can improve upon* s if there exist strategies $(t_i)_{i\in S} = t_S$ such that $p_i(s) < p_i(s_{-S}, t_S)$ for every $i \in S$. We say that the strategy profile s is a *strong Nash equilibrium* if s cannot be improved upon by any coalition (Aumann 1967).

Assume $s^* = (\lambda^*, x^*) \in \mathcal{S}$ is a strong Nash equilibrium of $\Gamma(v)$. To simplify notation, for any $i \in N$, let $\alpha_i(s^*) = \alpha_i^*$, $p_i(s^*) = p_i^*$, and $\mathcal{F}(s^*) = \mathcal{F}^*$.

Proposition 3.1 *If s^* is a strong Nash equilibrium of $\Gamma(v)$, then p^* is an aspiration of v.*

Proof. First, we show that $p^* = (p_i^*)_{i\in N}$ is aspiration feasible. For any $j \in N$, if $\alpha_j^* = 0$ then $p_j^* = v(\{j\}) = 0$ and thus p^* is aspiration feasible for j. If $\alpha_j^* > 0$, there must exist $S \in \mathcal{N}$ with $j \in S$ and $\lambda_0(S) > 0$. This implies that $p^*(S) \leq x^*(S) \leq v(S)$, and thus p^* is aspiration feasible.

We show next that p^* is stable. Suppose not, so $\exists S \in \mathcal{N}$ such that $p^*(S) < v(S)$. Define for every $i \in S$ $\hat{x}_i = p_i^* + \frac{\varepsilon}{|S|}$, where $\varepsilon = v(S) - p^*(S) > 0$. In this case S improves upon s^* by using strategies $(\mathbf{e}_S, \hat{x}_i)_{i\in S}$, which is a contradiction. We conclude that p^* is an aspiration of v. ■

Proposition 3.2 *If $s^* = (\lambda^*, x^*)$ is a strong Nash equilibrium of $\Gamma(v)$, then $p^* \in AC(v)$ and $\mathcal{F}^*$ is a balanced subset of $\mathcal{GC}(p^*)$. Conversely, if $y^* \in AC(v)$ then there exists a strong Nash equilibrium s^* of $\Gamma(v)$ with $p^* = y^*$ and $\mathcal{F}^* = \mathcal{GC}(y^*)$.*

Proof. Let s^* be a strong Nash equilibrium. According to Proposition 3.1, p^* is an aspiration of v. To show $p^* \in AC(v)$ it is therefore sufficient to prove that $\mathcal{GC}(p^*)$ is balanced. To do so, we show that $\mathcal{F}^* \subseteq \mathcal{GC}(p^*)$ and that $\mathcal{F}^*$ is itself balanced.

Recall that $\mathcal{F}^* = \{S \in \mathcal{N} | \lambda_0^*(S) > 0\} \cup \{\{i\} | \, \alpha_i^* < 1\}$. If $\lambda_0^*(S) > 0$, then x^* is feasible for S. As $p^*(S) \leq x^*(S)$ and p^* is an aspiration, $S \in \mathcal{GC}(p^*)$. If $S = \{i\}$ and $\alpha_i^* < 1$, we show that $p_i^* = v(\{i\}) = 0$. Assume $\alpha_i^*, x_i^* > 0$, otherwise the result holds trivially. If $p_i^* > 0$, there exists $T \ni i$ such that $\lambda_0^*(T) > 0$. As $\alpha_i^* < 1$ we have $p^*(T) = \sum_{j\in T} \alpha_j^* x_j^* < \sum_{j\in T} x_j^* \leq v(T)$, which is impossible as p^* is an aspiration. Therefore, $\mathcal{F}^* \subseteq \mathcal{GC}(p^*)$.

Finally, associate weights $\{\delta_S\}_{S\in\mathcal{F}^*}$ as follows. If $\lambda_0^*(S) > 0$ and $S \notin \{\{i\} |\ \alpha_i^* < 1\}$, let $\delta_S := \lambda_0^*(S)$. If $\alpha_i^* < 1$, let $\delta_{\{i\}} = 1 - \alpha_i^*$. Clearly, $\mathcal{F}^*$ is a balanced family of coalitions with the above balancing weights.

Conversely, let $y^* \in AC(v)$, $\mathcal{GC}(y^*)$ its corresponding balanced family of coalitions and $\{\delta_S\}_{S\in\mathcal{GC}(y^*)}$ some balancing weights. Consider the strategy profile $s^* = (\lambda^*, y^*)$ such that $\lambda_i^*(S) = \delta_S$ if $S \in \mathcal{GC}(y^*)$ and $i \in S$, and $\lambda_i^*(S) = 0$ otherwise. Clearly $\alpha_i^* = 1$ for every $i \in N$, so $p^* = y^*$. The strategy profile s^* is a strong Nash equilibrium because, if $\hat{s} = (\hat{\lambda}, \hat{y})$ improved upon s^* on $\hat{S}$, then $v(\hat{S}) \geq \hat{y}(\hat{S}) > y^*(\hat{S})$, so y^* would not be an aspiration.

■

Proposition 3.2 implies that the coalition structure $(\mathcal{F}^*, \delta)$ is stable and efficient. Therefore, the game delivers not only aspiration core allocations, but also the efficient coalition structures that support those allocations.

A similar strategic game was proposed by Kalai, Postlewaite, and Roberts (1979) in the context of a public good economy. They implement core allocations in strong Nash equilibrium. It is remarkable that a simple adaptation of their ideas allows us to draw important conclusions regarding coalition formation. This is the main achievement of this work.

A version of Kalai, Postlewaite, and Roberts's (1979) result can be obtained by applying Proposition 3.2 in the particular case the game v is balanced.

Corollary 3.3 *If v is balanced and s^* is a strong Nash equilibrium, then $p(s^*) \in C(v)$. Conversely, if $x^* \in C(v)$ then there exists a strong Nash equilibrium s^* with $p(s^*) = x^*$.*

The non-cooperative game above relied on the assumption that players can divide their resources/time among various coalitions. A natural question to ask is whether the aspiration core may still be used to predict coalition formation when agents' resources are assumed to be indivisible. To address this concern we define next a second procedure that does not require the co-existence of overlapping coalitions and, in equilibrium, also obtains the aspiration core.

4 The Bidding Game

The game described in this section illustrates the role of the aspiration core solution concept in the context of firm formation. It relates stability of aspirations to firm profitability and exploits the fact that coalition structures that support aspiration core allocations are precisely those that maximize the total surplus.

Given a cooperative TU game $v : 2^N \longrightarrow \mathbb{R}$, the following two-stage non-cooperative game, denoted by $\Psi(v)$, describes a situation in which an auxiliary set of individuals, called B_1, B_2 and X, compete over the n players in the cooperative game. X is the owner of a set of inputs (the elements of N) and sells them to potential entrepreneurs B_1 and B_2. Entrepreneurs will set up firms (i.e., "coalitions" of inputs) to produce the output. As in the previous section, we assume that v is zero-normalized without affecting the generality of our results. The description of the game is the following:

Stage 1: Principals B_1 and B_2 simultaneously bid amounts $b_1, b_2 \geq 0$. Let $b = \max_{i=1,2} b_i$ and label the winning bidder principal B. If bids are equal, a winner is selected at random. The losing bidder gets zero payoff and leaves the game.

Stage 2: Principals B and X engage in the following zero-sum game: B chooses a coalition $S \in \mathcal{N}$ and X chooses $x \in b\Delta_N$. Payoffs for B are determined according to the excess function $u_B(x, S) = v(S) - x(S)$. Payoffs for X are given by $u_X = -u_B$. Denote the zero-sum game by $\Omega(b)$.

The role of the first stage in this game is to pin down the total surplus achieved by the most efficient coalition structure. This step is not needed if the if the game v is balanced because in that case, the grand coalition is always the most efficient coalition structure. The second stage is related to other zero-sum games in the literature (see Aumann (1989), Gómez (2003), and Keiding and Pankratova (2010)).

A strategy profile for $\Psi(v)$ is $((b_1, \tau_1(b)), (b_2, \tau_2(b)), x(b))$ in which principal B_i bids $b_i \geq 0$ and, for any $b \geq 0$, chooses a mixed strategy $\tau_i(b) \in \Delta_{\mathcal{N}}$ in the second-stage game $\Omega(b)$. If the winning bid in stage 1 is b, Principal X chooses the vector $x(b) \in b\Delta_N$ in $\Omega(b)$. We now show that this non-cooperative procedure leads to the aspiration core vectors of game v.

We begin by analyzing the equilibria of the subgames that start after a winning bid b. The following proposition is a generalization of the min-max theorem to zero-sum semi-infinite games (i.e., games in which exactly one player has an infinite strategy set). We refer the reader to (Raghavan 1994) for a proof of this result.

Proposition 4.1 *For any $b \geq 0$ the semi-infinite zero-sum game $\Omega(b)$ has a value $\omega(b) \in \mathbb{R}$. Moreover, there exists a Nash equilibrium of $\Omega(b)$ in which*

X plays a pure strategy.

For ease of notation, denote the amount $\bar{b}(v)$ defined in (1) by $\bar{b}$.

Proposition 4.2 *The value $\omega(b)$ is a continuous and strictly decreasing function of b, with $\omega(\bar{b}) = 0$.*

Proof. That $\omega(b)$ is strictly decreasing follows immediately from the fact that $\max_{x \in b\Delta_N} \sum_{S \in \mathcal{N}} \lambda_S x(S)$ is strictly increasing in b, for every $\lambda \in \Delta_{\mathcal{N}}$. Continuity is an immediate consequence of Berge's maximum theorem (Border 1985, p. 64).

Since $\bar{b} = \max\{\sum_{S \in \mathcal{B}} \lambda_S v(S) \mid \mathcal{B}$ is balanced w.r.t. weights $\{\lambda_S\}\}$, there exists a balanced family $\mathcal{B}$ with weights $\{\bar{\lambda}_S\}$ such that $\sum_{S \in \mathcal{B}} \bar{\lambda}_S v(S) = \bar{b}$. Let $\bar{\Lambda} := \sum_{S \in \mathcal{N}} \bar{\lambda}_S$ and consider the mixed strategy $\bar{\tau}(b) \in \Delta_{\mathcal{N}}$ that assigns probability $\bar{\tau}_S(\bar{b}) := \frac{\bar{\lambda}_S}{\bar{\Lambda}}$ to $S \in \mathcal{B}$ and zero to any other $S \in \mathcal{N}$. Then, for any strategy $x \in \bar{b}\Delta_N$, we have

$$u_B(x, \bar{\tau}(\bar{b})) = \sum_{S \in \mathcal{B}} \frac{\bar{\lambda}_S}{\bar{\Lambda}} [v(S) - x(S)] = \frac{\bar{b} - x(N)}{\bar{\Lambda}} = 0.$$

Strategy $\bar{\tau}(\bar{b})$ guarantees Player B a non-negative profit, so $\omega(\bar{b}) \geq 0$. On the other hand, $\bar{b} = \min\{x(N) \mid v(S) \leq x(N) \ \forall S \subseteq N\}$ implies that $\exists \bar{x} \in \mathbb{R}^n$ such that $\bar{x}(N) = \bar{b}$ and $u_B(\bar{x}, S) \leq 0$ for every $S \subseteq N$. Consequently $\omega(\bar{b}) \leq 0$. We conclude $\omega(\bar{b}) = 0$, as we wanted. ■

Proposition 4.3 *The game $\Psi(v)$ has at least one subgame perfect Nash equilibrium.*

Proof. Proposition 4.1 implies the existence of at least one equilibrium

$(\hat{x}(b), \hat{\tau}(b))$ for every zero-sum game $\Omega(b)$. We claim that the strategy profile $((\bar{b}, \hat{\tau}(b)), (\bar{b}, \hat{\tau}(b)), \hat{x}(b))$ is an SPNE. Indeed, by definition, second-stage strategies are equilibria of $\Omega(b)$. Additionally, according to Proposition 4.2, no player can generate positive profits by deviating from his first stage strategy when the opponent bids $\bar{b}$. ■

Proposition 4.4 *In any SPNE, principals B_1 and B_2 choose bids equal to $\bar{b}$, yielding zero profits for Principals B_1, B_2 and X.*

Proof. Let $((b_1^*, \tau_1^*(b)), (b_2^*, \tau_2^*(b)), x^*(b))$ be an SPNE strategy profile for this game. Then $b_1^* = b_2^*$, otherwise, by strict monotonicity of $\omega(b)$, the highest bidder has an incentive to bid less. Let $b^* = b_1^* = b_2^*$. A bid $b^* > \bar{b}$ cannot be part of an equilibrium because, according to Proposition 4.2, $\omega(b^*) < 0$ and thus any bidder would prefer to stay out by biding 0. If $b^* < \bar{b}$, since $\omega(b^*) > 0$, any bidder has a profitable deviation in announcing $b^* + \varepsilon$ for $\varepsilon > 0$ small enough to guarantee $\omega(b^* + \varepsilon) > \frac{1}{2}\omega(b^*)$. Such $\varepsilon > 0$ always exists because $\omega(b)$ is continuous and $\omega(b^*) > 0$. Thus $b^* = \bar{b}$ and $\omega(b^*) = 0$. ■

Proposition 4.5 *If $((b^*, \tau_1^*(b)), (b^*, \tau_2^*(b)), x^*(b))$ is an SPNE for $\Psi(v)$ then $x^*(b^*) \in AC(v)$. Conversely, if $\bar{x} \in AC(v)$, then there exists an SPNE strategy profile $((b^*, \tau_1^*(b)), (b^*, \tau_2^*(b)), x^*(b))$ such that $x^*(b^*) = \bar{x}$.*

Proof. The previous proposition shows $b^* = \bar{b}$. By definition of $\bar{b}$ we know that any vector x such that $x(N) < \bar{b}$ cannot be an aspiration as it will be blocked by some coalition. It is then enough to show that $x^*(\bar{b})$, x^* for short, is an aspiration. If $x^*(S) < v(S)$ for some $S \in \mathcal{N}$ then the winning bidder could make positive profits choosing S, which would contradict Proposition 4.4. Thus, x^* is stable. Second, if there ia an $i \in N$ such that $x^*(S) > v(S)$ for every $S \ni i$, then coordinate x_i^* could be slightly reduced so that inequalities $x^*(S) \geq v(S)$ are still satisfied, and definition of $\bar{b}$ would be contradicted. Hence, x^* is also aspiration feasible and therefore an aspiration core allocation.

Conversely, let $\bar{x} \in AC(v)$. Let $b^* = \bar{b}$ and $(\tau(b), x(b))$ be a Nash equilibrium of the game $\Omega(b)$. For every $b \neq \bar{b}$ and $i = 1, 2$ define $\tau_i^*(b) = \tau(b)$. As $\mathcal{GC}(\bar{x})$ is balanced, denote its balancing weights by $\{\bar{\lambda}_S\}_S$. Let $\bar{\Lambda} = \sum_{S \in \mathcal{GC}(\bar{x})} \bar{\lambda}_S$. Define $\tau_{i,S}^*(\bar{b}) = \frac{\bar{\lambda}_S}{\bar{\Lambda}}$ if $S \in \mathcal{GC}(\bar{x})$ and zero otherwise. Finally, define $x^*(b) = x(b)$ for $b \neq \bar{b}$ and $x^*(\bar{b}) = \bar{x}$.

We show next that both Principals play a best response in the subgame $\Omega(\bar{b})$. For Principal B, giving positive weight to any $T \notin \mathcal{GC}(\bar{x})$ offers a lower payoff (excess), as $\bar{x}$ is an aspiration. For principal X, similar calculations to those performed in Proposition 4.2 show that $u_X(\tau^*(\bar{b}), x)$ is independent of $x \in \bar{b}\Delta_N$. Furthermore, we can now use the argument in Proposition 4.3 to show that players cannot benefit from choosing a bid different from $\bar{b}$. We conclude that the strategy profile $((b^*, \tau_1^*(b)), (b^*, \tau_2^*(b)), \bar{x})$ is an SPNE. ■

The game suggests a natural probabilistic interpretation of an overlapping coalition structure. Instead of claiming that non-disjoint coalitions form simultaneously, each coalition is assigned a positive probability of actually forming. We call this a *probabilistic coalition structure*. Ex-post, a single coalition forms. As shown bellow, the set of coalitions that have a positive probability of forming is a balanced family and the associated vector of probabilities is proportional to the vector of balancing weights.

Let $((\bar{b}, \tau_1^*(b)), (\bar{b}, \tau_2^*(b)), x^*(b))$ be an SPNE for $\Psi(v)$ and define

$$\mathcal{P}_k := \{S \mid \tau_k^*(\bar{b})(S) > 0\}$$

to be the associated family of *productive coalitions* (or *firms*) that can form, depending on the identity, $k = 1, 2$, of the winning bidder. Each firm $S \in \mathcal{P}_k$ forms with probability $\tau_k^*(\bar{b})(S)$. For every $k = 1, 2$, define $T_k^* := \max_{i \in N} \sum_{S \ni i} \tau_k^*(\bar{b})(S)$ and let

$$\mathcal{I}_k := \{i \in N \mid \sum_{S \ni i} \tau_k^*(\bar{b})(S) = T_k^*\}$$

be the set of players (machines) most likely to be chosen by the winning bidder B_k. For every $S \in \mathcal{P}_k$, define the participation coefficient $\lambda_S := \frac{\tau_k^*(\bar{b})(S)}{T_k^*}$. Clearly, for every $i \in N$, $\sum_{\mathcal{P}_k \ni S \ni i} \lambda_S \leq 1$, with equality if and only if $i \in \mathcal{I}_k$. If $N \setminus \mathcal{I}_k \neq \emptyset$, define $\lambda_{\{i\}} := 1 - \sum_{\mathcal{P}_k \ni S \ni i} \lambda_S$ for every $i \notin \mathcal{I}_k$ and let $\widetilde{\mathcal{P}}_k := \mathcal{P}_k \cup \{\{i\} | i \notin \mathcal{I}_k\}$.

Proposition 4.6 $(\widetilde{\mathcal{P}}_k, \lambda)$ *is an efficient and stable coalition structure. If* $x^*(\bar{b}) >> 0$ *then* $\mathcal{I}_k = N$.

Proof. According to Proposition 4.5, $x^* := x^*(\bar{b}) \in AC(v)$ and thus $v(S) \leq x^*(S)$ for all $S \subseteq N$. On the other hand, Proposition 4.2 implies that $\omega(\bar{b}) = \sum_{S \in \mathcal{N}} \tau_k^*(\bar{b})(S)(v(S) - x^*(S)) = 0$ and thus $\tau_k^*(\bar{b})(S) > 0$ only if $v(S) = x^*(S)$, which proves that $\mathcal{P}_k \subseteq \mathcal{GC}(x^*)$.

In addition, if $i \notin \mathcal{I}_k$ then, since $x^* \in \text{argmax}_{x \in \bar{b}\Delta_N} \sum_i x_i(\sum_{S \ni i} \tau_k^*(\bar{b})(S))$, it must be that $x_i^* = 0$ and thus $\{i\} \in \mathcal{GC}(x^*)$. This implies that $\widetilde{\mathcal{P}}_k \in$

$\mathcal{GC}(x^*)$ and, since $\widetilde{\mathcal{P}}_k$ is balanced by definition, $(\widetilde{\mathcal{P}}_k, \lambda)$ is an efficient and thus stable overlapping coalition structure. Moreover, if $x^* >> 0$ then agent X's maximization problem implies that $\mathcal{I}_k = N$ and thus $\mathcal{P}_k = \widetilde{\mathcal{P}}_k$. ■

There is a simple one-to-one and onto relationship between the set of efficient overlapping coalition structures as defined in Section 2 and the probabilistic coalition structures that assign each player in N equal probability of participating in a coalition. Indeed, if $(\mathcal{B}, \lambda)$ is an overlapping coalition structure, then this can be mapped into a probabilistic coalition structure $(\mathcal{B}, \pi)$ in which each coalition S have a probability $\pi_S := \frac{\lambda_S}{\Lambda}$ of forming, where $\Lambda := \sum_{S \in \mathcal{B}} \lambda_S > 0$. Reciprocally, if $(\mathcal{B}, \pi)$ is a probabilistic coalition structure with $\pi \geq 0$, $\sum_{S \in \mathcal{B}} \pi_S = 1$ and $\sum_{S \ni i} \pi_S = \sum_{S \ni j} \pi_S$ for every $i, j \in N$, $i \neq j$, then $(\mathcal{B}, \lambda)$ is an overlapping coalitions structure, where $\lambda_S := \frac{\pi_S}{\Pi}$, with $\Pi := \sum_{S \ni i} \pi_S$. The results above show that the equilibrium family $\mathcal{P}_k$ is exactly the family of coalitions that would be selected with a positive probability by a social planner who is concerned with maximizing the expected total surplus and who is fair to the players, in the sense of giving each of them an equal probability of participating in a formed coalition.

5 Concluding Remarks

This paper has proposed a generalization of two non-cooperative procedures leading to outcomes in the core and generalized their predictions to tackle coalition formation. Coalitions are created endogenously and the core, when non-empty, still arises as the unique equilibrium outcome. In different contexts, Morelli and Montero (2003), Sun, Trockel, and Yang (2008), and Zhou

(1994) discuss the need to determine the payoffs of a cooperative game without the assumption that a given coalition structure (e.g., the grand coalition) will form. The underlying premise of this line of research is that the payoff and coalition formation processes should occur simultaneously and take feedback from each other. Our two non-cooperative procedures exploit this idea and deliver, as their equilibrium outcomes, the aspiration core allocations and their supporting family of coalitions.

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