RANDOM VARIABLES

RVS ARE NUMERICAL OUTCOMES OF EXPERIMENTS

(S, ξ, P)

S = REAL LINE, COMPLEX PLANE, N-SPACE

ξ = subsets $\{ \underset{\sim}{x} \le x \}$

P = probability measure, parameterized by x, cdf

cdf $P(x)$

- $0 \le P(x) \le 1$
- $P(-\infty) = 0$, $P(+\infty) = 1$
- P monotonic nondecreasing
- $P(x) = \int_{-\infty}^{x} p(x)\,dx$
- often useful to work with $\overline{P}(x) = 1 - P(x)$

$P(x) = \Pr[\underset{\sim}{x} \le x]$

$\overline{P}(x) = \Pr[\underset{\sim}{x} > x]$

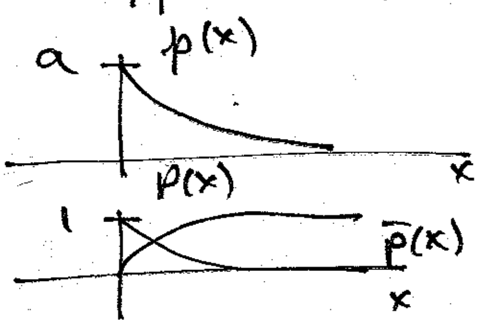
pdf $p(x)$

- $p(x) \ge 0$
- $\int_{-\infty}^{+\infty} p(x)\,dx = 1$
- $p(x) = \frac{d}{dx} P(x)$

$p(x)\,dx = \Pr[x < \underset{\sim}{x} \le x + dx]$

Ex: exponential distribution $p(x) = a e^{-ax} u(x)$

$a > 0$



check normalization

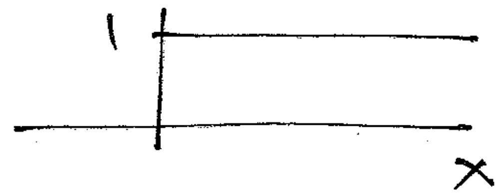
$$\int_{-\infty}^{+\infty} p(x)\,dx = 1$$

$$= \int_0^{\infty} a e^{-ax}\,dx = -e^{-ax}\Big|_0^{\infty} = 1$$

SOME USEFUL FUNCTIONS

- **UNIT STEP** $u(x)$

$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

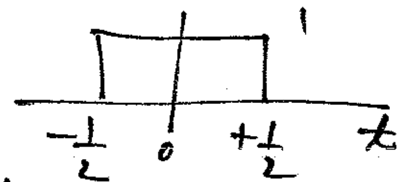


- **DELTA FUNCTION** (CT) $\delta(x)$

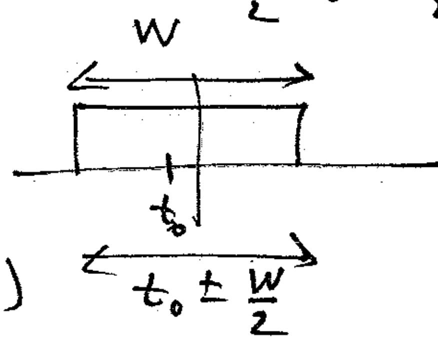
for a continuous $f(x)$

$$\int_{-\infty}^{\infty} \delta(x-x_0)\, f(x)\, dx = f(x_0)$$

- **RECTANGLE** $\text{RECT}(t)$



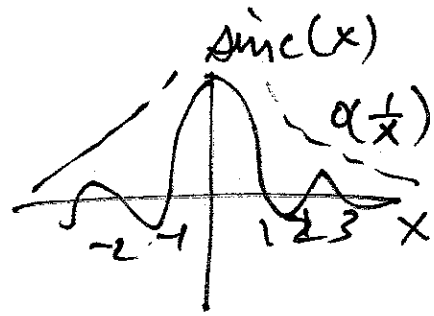
$$\text{RECT}\left(\frac{t-t_0}{W}\right)$$



t_0 = center of mass
W = TOTAL (2-sided WIDTH)

- **SINC** I DEFINE

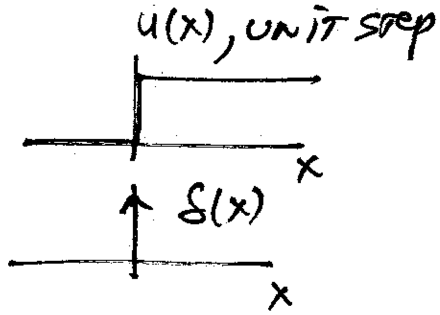
$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



- zero crossings at $x = \pm 1, \pm 2, \ldots$ (integers
- $\text{sinc}(0) = 1$ (use L'Hôpital's rule)

PDFS

DISCRETE RVS



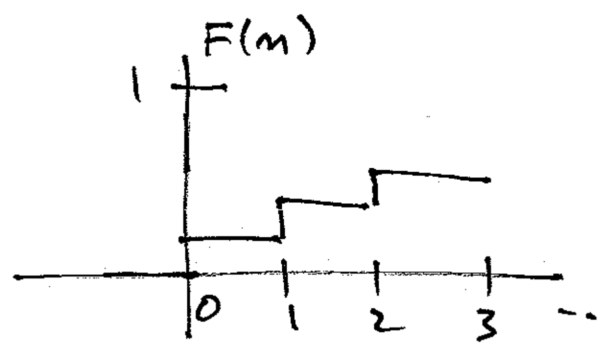
RECALL $\frac{d}{dx} u(x) = \delta(x)$

SINCE $\int_{-\infty}^{x} \delta(x)\,dx = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$

Consider the RV $\tilde{n}$, $\sum_{n=0}^{\infty} p_n = 1$

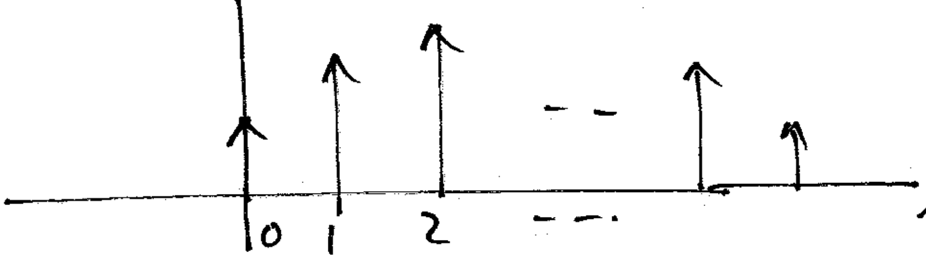
$p_n = \Pr[\tilde{n} = n] = \frac{\lambda^n e^{-\lambda}}{n!}$ for some $\lambda > 0$

POISSON RV, $n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$
$= n$ FACTORIAL



STAIRCASE

$f(n) = (d/dn)\, F(n) = \sum_{k=0}^{\infty} p_k\, \delta(k-n)$



we use the Dirac Delta to mark to locations of $\tilde{n}$.

CHECK $f(n) \geq 0$ ✓

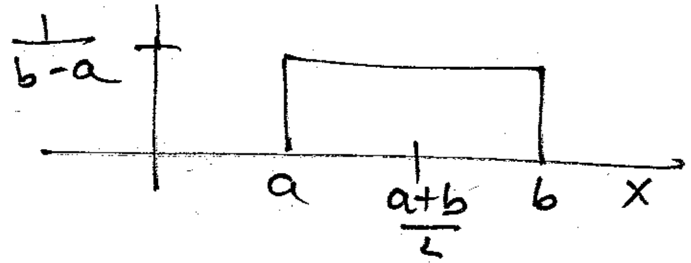
$\int_{-\infty}^{+\infty} f(n)\,dn = 1$

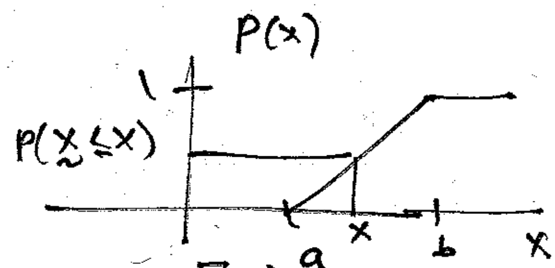
A RV whose pdf contains only δ's is called <u>discrete</u>.

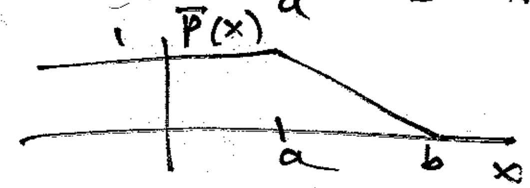
Poisson is very useful in telecommunications.

EX **UNIFORM DISTRIBUTION** $x \sim \text{unif}(a,b)$

$$p(x) = \frac{1}{b-a}\,\text{rect}\left(\frac{x-\frac{a+b}{2}}{b-a}\right) \qquad b > a$$

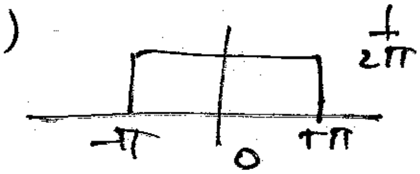






We say $\underset{\sim}{x}$ is uniformly distributed on $(a,b]$.

Typical example $\underset{\sim}{\theta} \sim \text{unif}(-\pi,+\pi)$ is a statistical model for a random phase angle.



EX: **GAUSSIAN OR NORMAL** $x \sim N(\mu,\sigma^2)$

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(x-\mu)^2/2\sigma^2}$$

$$P(x) = \int_{-\infty}^{x} p(x)\,dx = \int_{-\infty}^{x} e^{-\frac{(u-\mu)^2}{2\sigma^2}} \frac{du}{\sqrt{2\pi\sigma^2}}$$

let $t = \frac{u-\mu}{\sigma}$ $\qquad dt = \frac{du}{\sigma}$

$$P(x) = \Pr[\underset{\sim}{x} \le x] = \int_{-\infty}^{\frac{x-\mu}{\sigma}} \underbrace{e^{-t^2/2}\frac{dt}{\sqrt{2\pi}}}_{p(t)\,dt} = \text{erf}\left(\frac{x-\mu}{\sigma}\right)$$

here $\underset{\sim}{t} \sim N(0,1)$ since

$$p(t) = \frac{e^{-t^2/2}}{\sqrt{2\pi}}$$

$$\boxed{\text{So if } \underset{\sim}{t} \sim N(0,1) \quad \underset{\sim}{x} = \sigma t + \mu \sim N(\mu,\sigma^2)}$$

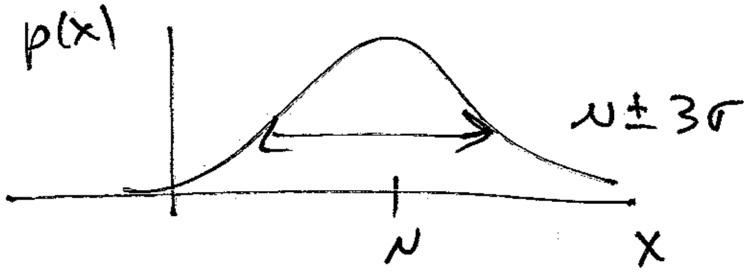
GAUSSIAN RV

A CONTINUOUS RV $\underset{\sim}{x}$ with pdf

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

is said to have $x \sim N[\mu, \sigma^2]$

A NORMAL OR GAUSSIAN DISTRIBUTION WITH mean μ and variance σ^2.



This is the most important pdf in signal processing.

DUE TO THE CENTRAL LIMIT THEOREM, THE BELL SHAPED CURVE IS MOST IMPORTANT IN STATISTICS.

$$\int_{-\infty}^{+\infty} p(x)\,dx = 1$$

$P(x) = \int_{-\infty}^{x} p(x)\,dx$ is a tabulated integral or can be done numerically.

is the cdf. (ERROR FUNCTION)

MATLAB: RANDN (NR,NC) GENERATES A SAMPLE OF $NR \cdot NC = N$ $N(0,1)$ RVS ORGANIZED IN A MATRIX OF NR ROWS, NC COLUMNS.

RAND (NR,NC) IS FOR UNIF (0,1) RVS.

- Examine some iid time series and histograms using matlab.
- Develop a plot of pdf, cdf and tabulated values using ERFC.M

MATHEMATICAL ASIDE ON THE GAUSSIAN

PROVE THAT
$$I = \int_{-\infty}^{+\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx = 1$$

PF. Let
$$I^2 = I \times I = \int_{-\infty}^{+\infty} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}$$
$$= \frac{1}{2\pi} \iint_{-\infty}^{+\infty} e^{-\frac{(x^2+y^2)}{2}}\, dx\,dy$$

Let $R^2 = x^2+y^2$ $\quad \tan(\theta) = y/x$

$dx\,dy = R\,dR\,d\theta$ IN POLAR COORDINATES

$$I^2 = \frac{1}{2\pi} \int_0^{\infty} \int_0^{2\pi} e^{-\frac{R^2}{2}}\, R\,dR\,d\theta$$
$$= \frac{1}{2\pi} \times 2\pi \times -e^{-R^2/2} \Big|_0^{\infty} = 1$$

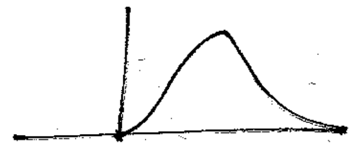
$\therefore\ I^2 = 1 \rightarrow \underline{I = 1}$ ✓

evaluate ① $\displaystyle\int_0^{\infty} e^{-t^2/2}\, dt = \frac{\sqrt{2\pi}}{2}$; ② $\displaystyle\int_{-\infty}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{dx}{\sqrt{2\pi\sigma^2}} = 1$

③ $\displaystyle\sqrt{\frac{\pi}{a}} = \int_{-\infty}^{+\infty} e^{-at^2}\, dt = \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2\sigma^2}} \frac{dt}{\sqrt{2\pi\sigma^2}}\ \sqrt{2\pi\sigma^2} = \sqrt{\frac{2\pi}{2}} = \sqrt{\frac{\pi}{a}}$ $\quad a = \frac{1}{2\sigma^2}$

④ $\displaystyle\int_0^{\infty} e^{-at^2}\, dt = \frac{1}{2}\sqrt{\pi/a}$

Rayleigh $\quad p(R) = \dfrac{R e^{-R^2/2\sigma^2}}{\sigma^2} u(R)$



$$P(R) = 1 - e^{-R^2/2\sigma^2}$$

$$\overline{P}(R) = e^{-R^2/2\sigma^2}$$

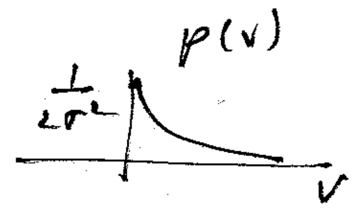
$$P(R) = \int_0^R r e^{-r^2/2\sigma^2} \frac{dr}{\sigma^2} \qquad \text{let } w = r^2, \quad dw = 2r\,dr$$

$$P(R) = \frac{1}{2\sigma^2} \int_0^{R^2} e^{-w/2\sigma^2} dw = \int_0^{R^2} a e^{-aw} dw \qquad a = \frac{1}{2\sigma^2}$$

[Recall exponential $a e^{-ax} u(x)$.]

$\therefore \quad Pr[\underset{\sim}{R} < R] = Pr[\underset{\sim}{V} < R^2]$

where $\underset{\sim}{V} \sim$ exponential



THIS SUGGEST $\quad \underset{\sim}{V} = \underset{\sim}{R}^2$

$\underset{\sim}{R} = \sqrt{\underset{\sim}{V}}$ WHICH IS TRUE.

AND WE NEED TO CONSIDER TRANSFORMATIONS OF IRV.

How does the pdf change?

If $\underset{\sim}{x} \sim (p, P, \overline{P})$ ← NOTATION

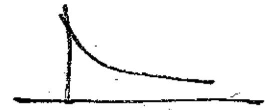
and $g(\cdot)$ is given, what is $(f, F, \overline{F})$

for $\underset{\sim}{y} = g(\underset{\sim}{x})$ $\quad$ g is the TRANSFORMATION.

EXAMPLES OF NORMALIZATION

EXPONENTIAL

(1) $f(t) = c\, e^{-at}$ $t > 0$



Then $\int_0^\infty f(t)\, dt = 1$

$$c \int_0^\infty e^{-at}\, dt = \frac{c}{a} = 1 \qquad \underline{c = a}$$

and $f(t) = a e^{-at}$ is A pdf

$E(t) = \frac{1}{a}$ (CHECK THIS)

(2) MIXTURES OF EXPONENTIALS

Let $a_1, a_2, \ldots a_k, \ldots a_M > 0$

and $f(t) = \sum_{k=1}^{M} w_k\, a_k\, e^{-a_k t}$

UNDER WHAT CONDITIONS OF $\{w_k\}$ IS $f(t)$ a pdf

(1) $\int_0^\infty f(t)\, dt = 1$ (2) $f(t) \geq 0$ everywhere

$$\sum_{k=1}^{M} w_k \int_0^\infty a_k e^{-a_k t}\, dt = \sum_{k=1}^{M} w_k = 1$$

$\therefore$ (i) $w_k \geq 0$ $k = 1, 2, \ldots M$

and (ii) $\sum_{k=1}^{M} w_k = 1$ THESE WEIGHTS MUST SUM TO 1

THE GAMMA FUNCTION

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt \quad ; \quad 0 < x$$

- GENERALIZES THE FACTORIAL TO NON INTEGER n

$$n! = \Gamma(n+1) = \int_0^\infty t^n e^{-t}\, dt$$

- VERY USEFUL IN PROBABILITY FOR NORMALIZATION INTEGRALS.
- M-FILE IN MATLAB GAMMA.M
- SIMILIAR TO A LAPLACE TRANSFORM $\{ \mathcal{L}[t^{x-1}] \}$
- EVALUATION FOR INTEGER $x-1 = n$

$$\int_0^\infty t^n e^{-t}\, dt = \int_0^\infty t^n e^{-st}\, dt \Big|_{s=+1}$$

(CHECK THIS YOURSELF)

$$= (-)^n \left(\frac{d}{ds}\right)^n \int_0^\infty e^{-st}\, dt \Big|_{s=+1}$$

$$= (-)^n \left(\frac{d}{ds}\right)^n \frac{(-)^n n!}{s^{n+1}} \Big|_{s=+1} = n!$$

- EXAMPLE IN NORMALIZATION

$$f(t) = C t^a e^{-t^b} \qquad t > 0 \qquad \text{DETERMINE } C$$

NORMALIZATION

$$C \Rightarrow \int_0^\infty f(t)\, dt = 1$$ and change variable

NORMALIZATION EXAMPLE

$$f(t) = c\,t^{a} e^{-t^{b}} \quad ; \quad t > 0 \quad , \quad f(t) \geq 0$$

$$\int_0^\infty f(t)\,dt = 1 \qquad \text{CONDITION FOR A pdf}$$

$$= c \int_0^\infty t^{a} e^{-t^{b}}\,dt \qquad \text{Let } u = t^{b}$$

$$du = b t^{b-1}\,dt$$

$$t = u^{\frac{1}{b}}$$

$$dt = \tfrac{1}{b} u^{\frac{1}{b}-1}\,du$$

$$= c \int_0^\infty u^{\frac{a}{b}} e^{-u} \tfrac{1}{b} u^{\frac{1}{b}-1}\,du$$

$$= \frac{c}{b} \int_0^\infty u^{\frac{a+1}{b}-1} e^{-u}\,du$$

$$\left[\text{since } \Gamma(x) = \int_0^\infty u^{x-1} e^{-u}\,du\right]$$

$$1 = \frac{c}{b}\,\Gamma\!\left(\frac{a+1}{b}\right)$$

SOLVE

$$c = \frac{b}{\Gamma\left(\frac{a+1}{b}\right)}$$

and

$$f(t) = \frac{b}{\Gamma\left(\frac{a+1}{b}\right)}\, t^{a} e^{-t^{b}} \qquad t > 0$$

is a pdf for $b > 0$

$$\frac{a+1}{b} \neq 0, -1, -2, \ldots$$

THIS IS CALLED THE <u>GENERALIZED GAMMA PDF</u>
AND USUALLY WRITTEN

$$f_{G\Gamma}(t; a, b, k) = C\left(\frac{t}{a}\right)^{bk-1} e^{-(t/a)^{b}} \qquad t > 0$$