FUNCTIONS of 1 RV.

Given: RV $\underset{\sim}{x}$ with pdf $f(x)$, cdf $F(x)$

function $g(x)$ (NON-RANDOM)

Problem: If $\underset{\sim}{y} = g(\underset{\sim}{x})$ (TRANSFORMATION)

WHAT IS THE PDF $p(y)$, CDF $P(y)$.

General Results.

For differentiable $g(x)$ with $g'(x) = \frac{d}{dx} g(x)$

$$p(y) = \sum_i \frac{f(x_i)}{|g'(x_i)|}$$

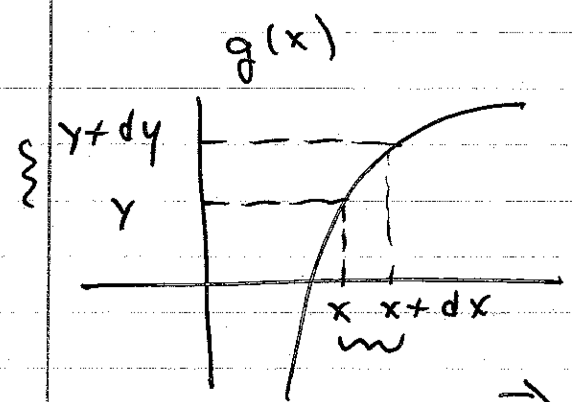
and sum over all solutions of $g(x_i) = y$.

ALWAYS have,

$$\underline{P}(y) = \Pr\{g(\underset{\sim}{x}) \le y\}$$

$$= \int u[-g(x) + y]\, f(x)\, dx$$

integrate over all x where $y - g(x) \ge 0$



$\Pr\{x < \underset{\sim}{x} \le x + dx\}$

$\equiv \Pr\{y < \underset{\sim}{y} < y + dy\}$

when $\underset{\sim}{y} = g(\underset{\sim}{x})$

$$\Rightarrow f(x)|dx| = p(y)\, |dy|$$

$$\Rightarrow p(y) = \sum \frac{f(x)}{|dy/dx|} \Big|_{x = g^{-1}(y)}$$

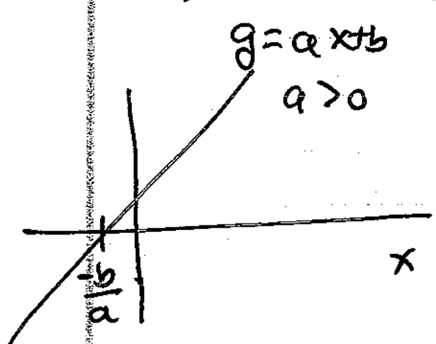
Important Examples.

LINEAR $\quad g(x) = ax+b \quad$ Here $g^{-1}(y) = \frac{y-b}{a}$ is unique (one-one map g)

Then $\{ g(\underset{\sim}{x}) \le y \}$ can be equated to simpler events

$$\{ax+b \le y\} = \begin{cases} \{x \le \frac{y-b}{a}\} & \text{if } a>0 \\ \{x > \frac{y-b}{a}\} & \text{if } a<0 \\ \{b \le y\} & \text{if } a=0. \end{cases}$$

Thus, $\quad P(y) = Pr\{\underset{\sim}{y} \le y\} = Pr\{a\underset{\sim}{x}+b \le y\}$



$$= \begin{cases} \int_{-\infty}^{\frac{y-b}{a}} f(x)\,dx & a>0 \\ \int_{\frac{y-b}{a}}^{\infty} f(x)\,dx & a>0 \end{cases}$$

$$\boxed{P(y) = \begin{cases} F\left(\frac{y-b}{a}\right) & a>0 \\ 1-F\left(\frac{y-b}{a}\right) & a<0 \end{cases}}$$

$$\boxed{p(y) = \frac{1}{|a|} f\left(\frac{y-b}{a}\right)}$$

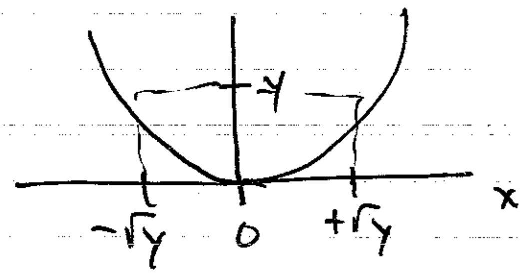
Specialize further for specific $f(x)$ or $F(x)$.

QUADRATIC

$$g(x) = x^2 \qquad g'(x) = 2x$$

$$g^{-1}(y) = \begin{cases} +\sqrt{y} \\ -\sqrt{y} \end{cases}$$

many x, yield 1 y.



$$P(y) = \Pr\{ \underset{\sim}{y} \le y \} = \Pr\{ \underset{\sim}{x}^2 \le y \}$$

$$= \Pr\{ -\sqrt{y} < \underset{\sim}{x} \le +\sqrt{y} \}$$

$$P(y) = \int_{-\sqrt{y}}^{+\sqrt{y}} f(x)\,dx$$

DIFFERENTIATE TO yield $p(y) = \dfrac{f(+\sqrt{y}) + f(-\sqrt{y})}{2\sqrt{y}}$

$$p(y) = \frac{d}{dy} P(y) = \frac{d}{dy} \int_0^{\sqrt{y}} f(x)\,dx + \frac{d}{dy} \int_{-\sqrt{y}}^{0} f(x)\,dx$$

$$= f(+\sqrt{y}) \frac{d\sqrt{y}}{dy} - f(-\sqrt{y}) \frac{d}{dy}(-\sqrt{y})$$

$$p(y) = \frac{f(+\sqrt{y})}{2\sqrt{y}} + \frac{f(-\sqrt{y})}{2\sqrt{y}}$$

LOGARITHMIC

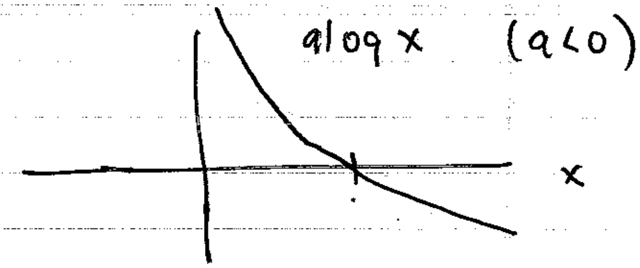
$$g(x) = a \log x \qquad (\log = \ln = \log_e)$$

$$g'(x) = \frac{a}{x} \qquad \text{here take } a<0$$

$P(y) = \Pr\{a \log \underset{\sim}{x} \le y\}$

$= \Pr\{\log \underset{\sim}{x} > \frac{y}{a}\}$

[note that $a<0$]



$$P(y) = \Pr\{\underset{\sim}{x} > e^{y/a}\}$$

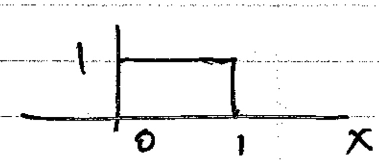
$$\boxed{P(y) = 1 - F(e^{y/a})}$$

$$p(y) = -\frac{1}{a} e^{y/a} f(e^{y/a})$$

$$p(y) = \frac{1}{|a|} e^{y/a} f(e^{y/a}) \qquad \text{either } \underline{y \lessgtr 0}$$

$$p(y) = \frac{f(x)}{|a/x|}\Bigg|_{x = e^{y/a}}$$

Special case when $\underset{\sim}{x} \sim \text{unif}(0,1)$



$$p(y) = \frac{x}{|a|} \cancel{f(x)}^{\to 1}\Bigg|_{x=e^{y/a}} \qquad 0 \le x \le 1 \text{ only. when } y \ge 0$$

$$= \frac{e^{y/a}}{|a|} = \frac{1}{|a|} e^{-y/|a|} \qquad \text{since } a<0.$$

EXPONENTIAL PDF.