Discrete Random Variables

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Random Variables

- RVs - numerical outcomes of random experiments

- Replace {tails, heads } by S = {0,1} |S|=2
- Replace {red, green, blue, yellow} by { 1, 2, 3, 4}

- Conceptually simple if |S| is finite or countable

- A discrete RV is a numerical outcome whose sample space S is finite or countable.
- Often conceptualized as a mapping from the Discrete Sample Space to a number, vector

Discrete RVs

- Bernoulli
- Binomial
- Uniform
- Geometric
- Poisson
- Negative Binomial

- Many others that are used in system modeling
- Wikipedia is a good source for probability distributions

Questions to Ask about RVs

- What is the experiment? Real-World Appls?
- How to generate in software? Simulation.
- What are the …
- Events? Outcomes? Probabilities?

- How do we compute the probabilities?
- How do we visualize the distribution?

- However, often RVs are composed/computed from simpler RVs through functions of RVs :

- Sum, product, min, max, kth largest of n, etc

Standard Terminology

- Discrete RV x in [1:N] boldface, uppercase, tilde
- Do not confuse the RV x with the possible outcomes x_1, x_2, …, x_N
- P_n = Pr(x=n) = PMF Probability Mass Function
- P_n >= 0   where \sum P_n = 1
- These normalization conditions are important constraints on the PMF P_n

Bernoulli (p)

- RV x in [0,1] = {heads, tails} or {success, failure}
- Pr[x=1] = p, then Pr[x=0] = q:= 1-p
- Normalization 0 <= p, q <= 1 where p +q = 1

- MATLAB: >> x= rand<q;% generates 1 Bern (p)

- Sketch the PMF here!
- Odds ratio is p/q

- The Bern (p) is a basic building block in modeling

Repeated Independent Trials

- Model – a sequence of T iid trials of Bern(p)
- Bernoulli Process
- iid := independent and identically distributed

- sequence of x= (x_1 … x_t) finite or infinite seq
- Each x_i is an iid Bern (p)
- The sample space is all 2^t binary strings.
- The probability of a binary string with k ones and t-K zeros i
- P(1…1…0…0) = p^k q^(t-k) same for all ordering
- >> rand(1,10)<.2
- ans = 0 0 1 1 0 1 0 0 0 0

Binomial (n, p)

- Consider a Bern(p) trail with n tosses.
- Outcome is the nx1 random sequence x

- Each element of x is 0 or 1, $P(x_i = 1) = p$

- The Binomial RV is the sum the sequence – total count of the ones RV $y = \sum x_i$
- Clearly $0 <= y <= n$ can show

- $P(y = k) = \{ n \text{ choose } k \} p^k q^{(n-k)}$

- Simulate by counting Bernoulli Trials

Derivation

- To prove this we look at all the outcomes that result in a total count of k out of n
- There are {n choose k} ways each with probability
- p^k q^(n-k)   p+q =1

- The probability of an event is the sum of the probabilities of all outcomes union totals the event of interest.R elabel the sample space!

- P(2 out of 3) = P(110) +P(011) +P(011)

Binomial Coefficients

- {n choose k} := n_C_k = n!/(n-k)! k!
- Much cancellation of common factors

- n_C_k = \prod_{j=0}^{k-1} [(n-j)/(1+j)]
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- Can compute n! = gamma(n+1)
- using MATLABs' gamma.m
- Asymptotic approximations –use Stirling's formula
- n! = \sqrt(2 \pi) n^(n+1/2) exp(-n) as n-> \inf

Computing Binomial (n, p)

- Cumulative Distribution Function CDF

- PMF $P(y = k) = \{ n \text{ choose } k \}\, p^k q^{(n-k)}$
- Can be computed directly, but easier to apply
- The Incomplete Beta Function betainc.m
- $I_x (a, b)$ = betainc(x,a,b)
- P(k) = P(***k*** <= k) = betainc(1-p, n-k, k+1);
- P(k) = P(***k*** > k) = betainc(p, k+1, n-k);

- Then P(a <= ***k*** <=b) = P(b)-P(a-1), etc
- Write out the math (don't rely on ppt!)

Computing Binomial (n, p)

- Cumulative Distribution Function CDF

- PMF P(y = k) = { n choose k } p^k q^(n-k)
- P(k) = P(***k*** <= k) = betainc(1-p, n-k, k+1);
- P(k) = P(***k*** > k) = betainc(p, k+1, n-k);
- n=7;p=0.77,q=1-p,k=5;
- binocdf(k, n, p),betainc(q,n-k,k+1)

- n=7;p=0.77,q=1-p,k=5;
- cdf1=binocdf(k, n, p),edf1=1-cdf1;betainc(p,k+1,n-k)
- Shows these are the same numerically

Waiting Times - Geometric

- Model: Flip coin repeatedly, mark the time until the first 1. RV x = number of tosses until first 1
- x = 1,2,3,4,5,... countable number of outcomes
- Unlimited number of trials Bern(p)

- To compute the probabilities, relabel the events!
- P(x=1) = P(1) = p, P(x=2) = P(01)=qp,
- P(x=3) = P(001) = q^2p P(x=4) = P(0^31) =q^3p

- P(x=n) =P_n=q^(n-1)p with \sumP_n =1 Geo(p)
- Here n=1,2,3,...
- This is a probabilistic proof of the geometric series

Computing Geometric (p)

- P(y = k) = q^(k-1)p, k=1,2,3,…
- Can be computed directly,
- using partial sums of geometric series
- Cumulative Distribution Function CDF
- P(k) = P(***k*** <= k) = \sum(j=[1:k]) q^(j-1)p
- This has a closed form!
- Alternatively use cumsum for numerics
- Write out the math (don't rely on ppt!)

Negative Binomial Distribution

- Again, observe an unlimited sequence of Bern(p)
- Repeated independent trials, iid
- Observe waiting time x until r^th 1 for any r=1,2,3

- Generalization of the Geometric distribution
- To find the distribution of P_n, look at n=0,1,2,…

- P(x<r) = 0,
- P(x=r) = P({0^(r-1) 1}) = q^(r-1) p
- P(x=r+1) – note the event can occur in several ways

Negative Binomial Derivation

- Think for a long time …

- How can the r^th 1 occur on the n^th trial???
- Last toss must be a 1 AND earlier tosses must have had r-1 ones in n-1 trials, in any order
- Put this together after recalling the binomial

- P(x=n) = {n-1 choose r-1} p^(r-1) q^(n-r) p
- Non-zero only for n = r, r+1, r+2,…
- Total Probability sums to 1

Computing Neg Bin (n,r,p)

- P(x=n) = {n-1 choose r-1} p^(r-1) q^(n-r)
- Can be computed directly, but easier to apply
- The Incomplete Beta Function betainc.m
- I_x (a, b) = betainc(x,a,b)
- P(k) = P(***k*** <= k) = betainc(p, r, k-r+1);k=r,r+1,...
- Cumulative Distribution Function CDF
- Then P(a <= ***k*** <=b) = P(b)-P(a-1)
- Write out the math (don't rely on ppt!)

Applications of Waiting Times

- You are searching for 1 particular key out of K
- You try at random, without keeping track of previous trys (bad idea?!).
- What is the probability that you find the right key on the n^th try?
- Geo(p) with p=1/K
- Compare that with a search without replacement
- Try one key, try the next, etc worst case is K
- Interesting to ask how many more trials should be expected with random search – Mean,Variance etc

Poisson Distribution

- Among the most important discrete distributions
- Limiting case of the binomial
- Let C_n,k := { n choose k}=n! / k! / (n-k)!
- Binomial(n,p) RV   0 <= k <= n
- P_k = C_n,k p^k q^(n-k) for k in [0:n]

- Poisson Limit   Let n -> infinity, while p -> 0
- Such that a = np stays finite and non-zero

- P_k = a^k exp(-a)/k!   K=0,1,2,3,...
- Total probability is 1 by the exponential power series
- Computation Incomplete Gamma Function gammainc.m

Applications

- When interest is in k out of n, for large n, small p
- Accidents, call arrivals, low-light level photoelectrons, cell counts
- Extends to time series and higher dimensions

- The most important distribution in classical telephony
- The Poisson Process is one of the most important Discrete Stochastic Processes

Birthday Problem Approximation

- Seek P_k that exactly k of n people will have
- the same birthday – large n=500, p=1/365
- Then a=np = 1.3699 matches exact to within 3 places for all k

- Stream of n symbols, p is probability of error per symbol. Then for large n, small p, P_k is the probability of k symbol errors

- The Poisson parameter a = np is a unitless, a ratio of 2 numbers in the same units Why

More Poisson

- Now observe a time interval [0,T], with random incidents (photons, calls, failures, jobs,etc)
- Under a model in which the chance of overlapping incidents is small and non-overlapping incidents are independent, the Poisson Distribution arises

- Here a = rT where r= is the rate (per unit time)

- P_n = a^n exp(-a)/n! Note that a =rT is unitless,
- So that r is a rate of arrival per unit time

Poisson Distribution

- Poisson Limit   Let n -> infinity, while p -> 0
- Such that a = np stays finite and non-zero

- P_k = a^k exp(-a)/k!   k=0,1,2,3,...

- Computation Incomplete Gamma Function gammainc.m
- K~Poiss(a), P(k <=k) = 1-gammainc(k+1,a)
- Or cumsum: P(k <= k) =sum_{k=0:k} a^k exp(-a)/k!

- a=17;k=22;cdf1 = poisscdf(k,a),
- Cdf2 = 1-gammainc(a,k+1)

Aside: Unitless Quantities

- From physics recall formulas
- x = vt +at^2/2, velocity v, acceleration a, time t

- Clearly vt and at^2 must have the same units to be added with meaning (you can't add apples and oranges)

- Consider exp(x) = 1 + x+x^2/2! + …

- The same reasoning shows that x must be unitless
- Else x and x^2 do not have the same units

Probabilistic Proofs of Power Series

- By Total Probability (\sumP_n =1) we have proved

- \sum_{k=0:n} { n choose k} p^k q^(n-k) = 1
- \sum_{k>=1} q^{k-1}p = 1
- \sum_{k>=r} { n-1 choose r-1 } p^r q^(n-r) = 1

- Probability as an alternative to analysis …

References

- MATLAB help for betainc.m gammainc.m
- Feller An Intro to Probability Theory and Its Applications, vol. I
- CW Helstrom, Probability and Stochastic Processes for Engineers
- Grimmett and Stirzaker, Probability and Random Processes