EE 416

Midterm Exam, Autumn 2009

Wednesday Nov. 4, 2009, 10:30-12:20am

Name (Last, First):

1. **Problem 1:**
2. **Problem 2:**
3. **Problem 3:**
4. **Problem 4:**
5. **Problem 5:**
6. **Problem 6:**

   Score:

Instructions:

Boldface type (mostly) indicates a RV, $\mathbf{x}$, $\mathbf{t}$
Closed Book, 3 pages personal notes.No calculators, or electronic devices.
Be neat, *show* your work, *box* your answer, and *staple* your exam.

1. (20 points).

A joint distributed discrete random variable $(\mathbf{x}, \mathbf{y})$ takes value in $\mathbf{x} \in \{0, 1\}$, $\mathbf{y} \in \{0, 1, 2\}$, with probability:

	$x = 0$	$x = 1$
$y = 0$	0.12	0.34
$y = 1$	0.21	0.22
$y = 2$	0	0.11

For example, $\Pr(x = 0, y = 2) = 0$.

(5pts) Find the marginal distribution of $\mathbf{x}$.

$\Pr(x = 0) = 0.12 + 0.21 + 0 = 0.33$,

$\Pr(x = 1) = 0.34 + 0.22 + 0.11 = 0.67$.

(5pts) Find the marginal distribution of $\mathbf{y}$.

$\Pr(y = 0) = 0.12 + 0.34 = 0.46$,

$\Pr(y = 1) = 0.21 + 0.22 = 0.43$,

$\Pr(y = 2) = 0 + 0.11 = 0.11$.

(10pts) Find the conditional distribution of $\mathbf{y}$, given $\mathbf{x} = 1$. Leave the result as a fraction.

$\Pr(\mathbf{y} = i|\mathbf{x} = 1) = \frac{\Pr(x=1, y=i)}{\Pr(x=1)}, \quad i = \{0, 1, 2\}$.

$\Pr(\mathbf{y} = 0|\mathbf{x} = 1) = \frac{34}{67}$,

$\Pr(\mathbf{y} = 1|\mathbf{x} = 1) = \frac{22}{67}$,

$\Pr(\mathbf{y} = 2|\mathbf{x} = 1) = \frac{11}{67}$.

2. (20 points).
A general Poisson random variable has PMF

$$P_k = P\{\mathbf{k} = k\} = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, \cdots, \quad \lambda > 0$$

(10pts) You are given that $\Pr(\mathbf{k} = 0) = p_0$. Determine a formula for λ in terms of p_0.

$\Pr(\mathbf{k} = 0) = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-\lambda} = p_0$, thus $\lambda = -\ln p_0$.

(10pts) Find $\Pr(\mathbf{k} \geq 10 | \, \mathbf{k} \geq 5)$. Leave the result as a sum, there is no closed form.

$\Pr(\mathbf{k} \geq 10 | \, \mathbf{k} \geq 5) = \frac{\Pr(\{\mathbf{k} \geq 10\} \cap \{\mathbf{k} \geq 5\})}{\Pr(\mathbf{k} \geq 5)} = \frac{\Pr(\mathbf{k} \geq 10)}{\Pr(\mathbf{k} \geq 5)} = \frac{\sum_{k=10}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!}}{\sum_{k=5}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!}}$

or $\Pr(\mathbf{k} \geq 10 | \, \mathbf{k} \geq 5) = \frac{1 - \sum_{k=0}^{9} \frac{\lambda^k e^{-\lambda}}{k!}}{1 - \sum_{k=0}^{4} \frac{\lambda^k e^{-\lambda}}{k!}}$

3. (10 points).

A communications system can utilize different channels for different communicating users. A total of n channels are available, that k users seek to use. Each of the k channel assignments are selected randomly from among the n available, *with replacement*. That is, the users are uncoordinated. When two or more users occupy the same channel, a collision occurs.

(5pts) Give a general expression for $P(n,k)$, the probability of a collision, in terms of n, k.

Method 1:

There are n^k possibe outcome of channel assignments. There are $P_k^n = \frac{n!}{(n-k)!}$ outcomes of no collision.

Therefore, $\Pr(\text{collision}) = 1 - \Pr(\text{no collision}) = 1 - \frac{n!}{(n-k)!n^k}$.

Method 2:

Pr(no collision)

$$=\Pr\begin{pmatrix} \text{the 1st user chooses a channel from } n \text{ channels,} \\ \text{the 2nd user chooses a channel from the remaining } n-1, \\ \vdots \\ \text{the } k\text{-th user chooses a channel from the remaining } n-k+1. \end{pmatrix}$$

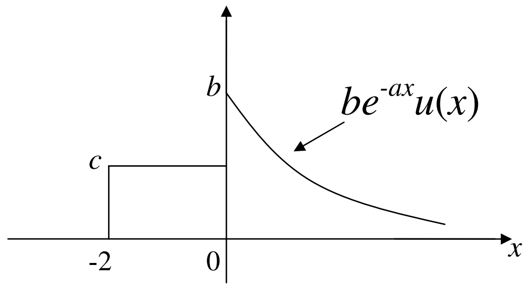
$= \frac{n}{n}\frac{n-1}{n}\cdots\frac{n-k+1}{n}$,

$\Pr(\text{collision}) = 1 - \Pr(\text{no collision})$.

(5pts) Numerically, determine $P(n=50, k=3)$, to two decimal places. No calculators are available; estimate.

$P(n=50, k=3) = 1 - \frac{50\times49\times48}{50\times50\times50} \approx 0.06$.

4. (25 points).



(10pts) Find all the conditions on c, a, b so that this is a valid pdf.

condition 1:

pdf has to be non-negative, Thus $c \geq 0$, $b \geq 0$.

condition 2:

The integral $\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x$ is 1.

$\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x = \begin{cases} 2C, & b = 0 \\ 2C + \int_0^{\infty} be^{-ax}\mathrm{d}x, & b > 0 \end{cases}$.

If $b = 0$, then $2C = 1$ and a can take any value;

If $b > 0$, then for the integral to converge a must satisfy $a > 0$, and then $2C + \int_0^{\infty} be^{-ax}\mathrm{d}x = 2C + \frac{b}{a} = 1$.

For the 2nd and 3rd question, we only consider $b > 0$, because the case of $b = 0$ is trivial.

(10pts) Compute $\Pr(\mathbf{x} > 1/a)$ in a closed form (not an integral).

Since $\frac{1}{a} > 0$, $\Pr(\mathbf{x} > 1/a) = \int_{\frac{1}{a}}^{\infty} be^{-ax}\mathrm{d}x = \frac{b}{a}e^{-1}$.

(5pts) Determine the $\mathrm{E}(\mathbf{x}|\,\mathbf{x} > 1/a)$.

Method 1:

conditional CDF is

$\Pr(\mathbf{x} \leq x|\,\mathbf{x} > 1/a)$

$= \frac{\Pr(\mathbf{x} \le x, \mathbf{x} > 1/a)}{\Pr(\mathbf{x} > 1/a)}$

$= \begin{cases} \frac{\Pr(1/a < \mathbf{x} \le x)}{\Pr(\mathbf{x} > 1/a)}, & x \ge \frac{1}{a} \\ 0, & x < \frac{1}{a} \end{cases}$

$= \begin{cases} \frac{\frac{b}{-a}\left(e^{-ax} - e^{-1}\right)}{\frac{b}{a}e^{-1}}, & x \ge \frac{1}{a} \\ 0, & x < \frac{1}{a} \end{cases}.$

conditional PDF is

$f_X(x|\,\mathbf{x} > 1/a)$

$= \frac{\mathrm{d}\Pr(\mathbf{x} \le x|\,\mathbf{x} > 1/a)}{\mathrm{d}x}$

$= \begin{cases} \frac{be^{-ax}}{\frac{b}{a}e^{-1}}, & x \ge \frac{1}{a} \\ 0, & x < \frac{1}{a} \end{cases}.$

$\mathrm{E}(\mathbf{x}|\,\mathbf{x} > 1/a) = \int_{-\infty}^{\infty} x \cdot f_X(x|\,\mathbf{x} > 1/a)\mathrm{d}x = \int_{\frac{1}{a}}^{\infty} x \cdot \frac{be^{-ax}}{\frac{b}{a}e^{-1}}\mathrm{d}x = \frac{2}{a}.$

Method 2:

By the memoryless property of the exponential random variable, if $\mathbf{y}$ is an exponential r.v., then $\mathbf{y}|\mathbf{y} > t$ has the same pdf as $t + \mathbf{y}$.

For $x > 0$, $\mathbf{x}$ is exponential distributed, then $\mathbf{x}|\mathbf{x} > t$ is equivalent to $t + \mathbf{y}$, where $\mathbf{y}$ is an exponential r.v. with rate $\lambda = a$.

Therefore, $\mathrm{E}(\mathbf{x}|\,\mathbf{x} > 1/a) = \mathrm{E}(1/a + \mathbf{y}) = 1/a + \mathrm{E}(\mathbf{y}) = 2/a$.

5. (15 points).

A Gaussian random variable has mean μ, and variance σ^2. The PDF is given by

$$p(x) = \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}.$$

(10pts) Write $\Pr(\mathbf{x} > x_0)$ in terms of

$$Q(x) = \int_x^\infty \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,\mathrm{d}t.$$

$\Pr(\mathbf{x} > x_0)$

$= \int_{x_0}^\infty \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\mathrm{d}x$

$= \int_{x_0-\mu}^\infty \frac{e^{-y^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\mathrm{d}y, \quad \text{let } y = x - \mu$

$= \int_{\frac{x_0-\mu}{\sigma}}^\infty \frac{e^{-t^2/2}}{\sqrt{2\pi}}\mathrm{d}t, \quad \text{let } t = \frac{y}{\sigma}$

$= Q\left(\frac{x_0-\mu}{\sigma}\right)$

(5pts) Determine the value of $Q(x = 0)$.

$\frac{e^{-t^2/2}}{\sqrt{2\pi}}$ is the pdf for Gaussian random variable with mean 0 and variance 1. Thus $\int_{-\infty}^\infty \frac{e^{-t^2/2}}{\sqrt{2\pi}}\mathrm{d}t = 1$.

By symmetry, $Q(x = 0) = \int_0^\infty \frac{e^{-t^2/2}}{\sqrt{2\pi}}\mathrm{d}t = \frac{1}{2}\int_{-\infty}^\infty \frac{e^{-t^2/2}}{\sqrt{2\pi}}\mathrm{d}t = \frac{1}{2}$.

6. (10 points).

A Laplace random variable $\mathbf{x}$ has PDF

$$p(x) = Ce^{-|x|}, \ -\infty < x < \infty.$$

(5pts) Determine C so that $p(x)$ is a valid pdf.

$\int_{-\infty}^{\infty} p(x)\mathrm{d}x = 1$. Thus $C = \frac{1}{2}$.

(5pts) Determine the PMF of

$$\mathbf{y} = \mathrm{sign}(\mathbf{x}).$$

If $\mathbf{x} < 0$, then $\mathbf{y} = -1$;

If $\mathbf{x} \geq 0$, then $\mathbf{y} = 1$.

Therefore $\mathbf{y}$ is a discrete random variable, taking values in $\{-1, 1\}$.

$\Pr(\mathbf{y} = -1) = \Pr(\mathbf{x} < 0) = \int_{-\infty}^{0} p(x)\mathrm{d}x = \frac{1}{2}$;

$\Pr(\mathbf{y} = 1) = \Pr(\mathbf{x} \geq 0) = \int_{0}^{\infty} p(x)\mathrm{d}x = \frac{1}{2}$.