EE 416

Final Exam, Autumn 2009

Monday Dec. 14, 2009,
8:30-10:20 AM

Name (Last, First):

Scoring

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	20	
6	10	
7	10	

Instructions:

Boldface type (mostly) indicates a RV, $\mathbf{x}$, $\mathbf{t}$. Closed Book, 110 minutes, 6 (two-sided) pages personal notes..
Be neat, *show* your work , and *staple* your exam.
Explain your method for partial credit

Problem 1 (15pts)

A RV $\mathbf{x}$ on$(-\infty, +\infty)$ has pdf

$$p(x) = \frac{1}{2}\exp(-|x-1|) \tag{1}$$

Evaluate $P[-1 \le \mathbf{x} \le +3]$
Evaluate $E[\mathbf{x}]$.

$$\begin{aligned}
P[-1 \le \mathbf{x} \le +3] &= \int_{-1}^{3} p(x)\,\mathrm{d}x \\
&= 2\int_{1}^{3} p(x)\,\mathrm{d}x \quad \text{(by symmetry)} \\
&= 2\int_{1}^{3} \frac{1}{2}\exp(-|x-1|)\,\mathrm{d}x \\
&= \int_{1}^{3} \exp(-(x-1))\,\mathrm{d}x \\
&= 1 - e^{-2}
\end{aligned}$$

$$E[\mathbf{x}] = 1 \quad \text{(by symmetry)}$$

Problem 2 (15pts)
A Laplace random variable $\mathbf{y}$ has pdf

$$p(y) = C\exp(-|y|) \tag{2}$$

What is C?

You are asked to design a quantizer that splits the domain of $p(y)$ into 4 segments. Each segment should have the same probability 1/4 of containing $\mathbf{y}$.

Find the 3 segments boundaries $[\ Y_1\ \ Y_2\ \ Y_3]$.

For $p(y)$ to be a valid pdf, $\int_{-\infty}^{\infty} p(y)\,\mathrm{d}y = 2C = 1$, then $C = \frac{1}{2}$.
The 1st segment is $[-\infty, Y_1]$, $\int_{-\infty}^{Y_1} p(y)\,\mathrm{d}y = \frac{1}{2}e^{-Y_1} = \frac{1}{4}$, then $Y_1 = -\ln 2$;
The 4th segment is $[Y_3, \infty]$, $\int_{Y_3}^{\infty} p(y)\,\mathrm{d}y = \frac{1}{2}e^{Y_3} = \frac{1}{4}$, then $Y_3 = \ln 2$;
For the probability in $[Y_1, Y_2]$ and $[Y_2, Y_3]$ to be $\frac{1}{4}$, $Y_2 = 0$ by symmetry.
Therefore, $[Y_1, Y_2, Y_3] = [-\ln 2, 0, \ln 2]$.

Problem 3 (15pts)
A system puts out a random IID sequence of $0, 1$ (ZERO, ONE) bits, with equal probabilities ($P(0) = P(1) = \frac{1}{2}$).

After n bits ($n = 1, 2, 3, \ldots$), find (as a function of n) the probability that the number of ONES equals the number of ZEROS?

For odd n, Pr[#ONES = #ZEROS]=0;

For even n, Pr[#ONES = #ZEROS]=$\begin{pmatrix} n \\ \frac{n}{2} \end{pmatrix} (\frac{1}{2})^{\frac{n}{2}} (\frac{1}{2})^{\frac{n}{2}}$;

Problem 4 (15pts)
Define $Q(x) := P[\mathbf{x} > x]$, when $\mathbf{x}$ is a zero-mean, unit variance, Gaussian RV, $N(0, 1)$. In terms of Q, find

$$P[-2\mathbf{x} + 3 < 0]$$

How would you compute this in Matlab?

$$\begin{aligned} P[-2\mathbf{x} + 3 < 0] &= P\left[\mathbf{x} > \frac{3}{2}\right] \\ &= Q\left(\frac{3}{2}\right) \\ &= \frac{1}{2}\left(\text{erfc}\left(\frac{3}{2\sqrt{2}}\right)\right) \\ &= \frac{1}{2}\left(1 - \text{erf}\left(\frac{3}{2\sqrt{2}}\right)\right) \end{aligned}$$

MATLAB code:

```
>> 1/2*erfc( 3/2/sqrt(2) )
```

or

```
>> 1/2*( 1-erf(3/2/sqrt(2)) )
```

Problem 5 (20pts)
A 2×1 random vector $\mathbf{x} = [x_1 \;\; x_2]^T$ has mean zero and covariance matrix (in terms of ρ)

$$\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

What is the allowable range of ρ?
Evaluate $E[(x_1 + x_2)^2]$

From the covariance matrix, $\sigma_{x_1} = 1$, $\sigma_{x_2} = 1$, so $\rho = \rho_{x_1,x_2}\sigma_{x_1}\sigma_{x_2} \in [-1, 1]$, where the correlation coefficient $\rho_{x_1,x_2} \in [-1, 1]$.

$E[(x_1 + x_2)^2] = E[x_1^2 + 2x_1x_2 + x_2^2]$
$= E[x_1^2] + E[2x_1x_2] + E[x_2^2]$
$= (\sigma_{x_1}^2 + \mu_{x_1}^2) + 2(cov(x_1, x_2) + \mu_{x_1}\mu_{x_2}) + (\sigma_{x_2}^2 + \mu_{x_2}^2)$
$= 1 + 2\rho + 1$
$= 2 + 2\rho$

Problem 6 (10pts)

Consider the following snippet of MATLAB code.

```
>> N=100;h = ones(N,1);
>> w = 2*randn(N,1);
>> x = conv(h,w);
```

How long will the output vector `x` be? The output will consist only of a noise component, due to the input noise `w`. What is the output noise variance in the steady state? You can neglect any filter transients. The effect of transients was never discussed in class.

The `x` has length 100+100-1=199.
The output at steady state is $\tilde{\mathrm{w}}=\mathbf{h}_{\text{flipped}}^T\mathtt{w}=\mathtt{h}(N)\mathtt{w}(1)+\cdots+\mathtt{h}(1)\mathtt{w}(N)=\mathtt{w}(1)+\cdots+\mathtt{w}(N)$, where $\mathtt{w}(i)$ has variance 4. Since $\mathtt{w}(i)$ are independent, $\mathrm{var}(\tilde{\mathrm{w}})=N\mathrm{var}(\mathtt{w}(i))=400$.

Problem 7 (10pts)

The $\mathbf{x}$ and $\mathbf{y}$ are two independent exponential random variables. We are given that $E[\mathbf{x}] = 1$, $E[\mathbf{y}] = 2$, so they are not identically distributed.

Compute $P[\mathbf{x} > \mathbf{y}]$.

method 1:
$\lambda_x = 1$, $\lambda_y = \frac{1}{2}$.
Joint probability $p(x, y) = \lambda_x e^{-\lambda_x x} \lambda_y e^{-\lambda_y y} = \frac{1}{2} e^{-x-\frac{1}{2}y}$.
$P[\mathbf{x} > \mathbf{y}] = \int_0^\infty \int_y^\infty \frac{1}{2} e^{-x-\frac{1}{2}y}\, \mathrm{d}x \mathrm{d}y$
$= \int_0^\infty \frac{1}{2} e^{-\frac{3}{2}y}\, \mathrm{d}y$
$= \frac{1}{3}$.

method 2 (property of exponential r.v., see wikipedia):
$P[\mathbf{x} > \mathbf{y}] = P[\mathbf{y} = \min(\mathbf{x}, \mathbf{y})] = \frac{\lambda_y}{\lambda_x + \lambda_y} = \frac{1}{3}$.