1 Continuous RV (10 pts)

Consider a positive random variable $\mathbf{t} > 0$. You know that

$$P[\mathbf{t} > t] = c(a - t), \quad 0 \le t \le a$$

1. (5pts) Let $a > 0$, What values of c lead to a valid distribution.
2. (5pts) Determine the pdf $p(t)$ of RV $\mathbf{t}$ in terms of a alone.

1. $F(t) = P[\underset{\sim}{t} \le t] = 1 - P[\underset{\sim}{t} > t] = 1 - c(a-t) \quad 0 \le t \le a$

$\lim_{t \to 0} F(t) = 0$

$\lim_{t \to a} F(t) = 1$

← property of CDF

$\Rightarrow \quad c = \frac{1}{a}$

2. $p(t) = \frac{dF(t)}{dt} = \frac{1}{a} \quad 0 \le t \le a$

so this is a unif(0, a)

2. K out of N Detection (15pts)

When a target is present, a radar detects it with probability $p_0 \approx 0.9$; if it is not present it falsely detects with probability $q_0 \approx= 0.01$. These are defined to be the *single-look detection* p_0 *and false alarm* q_0 *probabilities.* Note that p_0, q_0 are completely unrelated; for example, $q_0 + p_0 \neq 1$ in general.

The radar takes multiple looks and detects independently on each look. The probabilities remain fixed at the single-look values. Two different methods are used to determine whether the target is present on multiple looks, and we seek to compare these methods. Take N and K to be arbitrary such that

$$1 \leq K \leq N$$

In the questions, we will only consider the detection probability.

Part 1. If the radar takes $n = N$ looks, and decides on an overall detection using the rule:
Rule 1: A target is said to be Detected if it is detected on any single look.
what is the resulting probability of detection $P_D^{(1)}$. An exact, non-numeric symbolic answer in terms of p_0 is required.

1. (5pts) What is the probability $P_D^{(1)}$

   Part 2 The radar again looks $n = N$ times, but now votes:
   Rule 2: A target is said to be Detected if it is detected on at least K out of N looks .
   What is the resulting probability of detection $P_D^{(K)}$. Again, an exact non-numeric symbolic answer in terms of p_0 is required.

2. (5pts) What is the probability $P_D^{(K)}$

3. (5pts) For what values of K does the overall detection probability drop under Rule 2 compared with Rule 1? Be precise.

Part 1:

1. $P_D^{(1)} = P[\text{detected on any single look}]$
$= 1 - P[\text{not detected by any look}]$
because → $= 1 - (1-p_0)^N = 1 - (0.1)^N$
detects independently on each look

Part 2:

2. $P_D^{(K)} = P[\text{detected on ONLY K out of N looks}]$
$+ P[\text{detected on ONLY K+1 out of N looks}]$
$+ \dots + P[\text{detected on ALL N out of N looks}]$

$$= \sum_{i=k}^{N} \binom{N}{i} p_0^{i} (1-p_0)^{(N-i)}$$

$$= \sum_{i=k}^{N} \binom{N}{i} (0.9)^{i} (1-0.9)^{(N-i)}$$

3. for any value $K \geq 2$,
the overall detection probability of Rule 2
is lower than Rule 1.

3 Bernoulli (15pts)

A Bernoulli generator puts out $\{x_1, x_2, \ldots, x_n\}$ where $x_i \in \{0,1\}$ iid with $P(x_i = 1) = p$.

A second Bernoulli generator puts out $\{y_1, y_2, \ldots, y_n\}$ where $y_i \in \{0,1\}$ iid with $P(y_i = 1) = r$, independent of x_i

For any $k = 0, 1, 2, \ldots$

1. (10 points) Determine the probability that

$$\sum_{i=1}^{n} x_i = k$$

2. (5 points) Determine the probability that

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

1. $P[\sum_{i=1}^{n} x_i = k] = P[$k out of n x_i's are equal to one, others are equal to zero$]$

$$= \binom{n}{k} p^k (1-p)^{(n-k)}$$

2. $\sum_{i=1}^{n} x_i$ ranges from 0 to n

↑ all '0' ↑ all '1'.

$$P(\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i) = \sum_{k=0}^{n} P(\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = k)$$

because x & y are independent →

$$= \sum_{k=0}^{n} P(\sum_{i=1}^{n} x_i = k) \cdot P(\sum_{i=1}^{n} y_i = k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{(n-k)} \binom{n}{k} r^k (1-r)^{(n-k)}$$

$$= \sum_{k=0}^{n} \left[\binom{n}{k}\right]^2 (pr)^k \left[(1-p)(1-r)\right]^{(n-k)}$$

4 The Lost Signal (15 pts)

A signal is located in exactly 1 of N frequencies $F_1, F_2, \ldots, F_N$ with corresponding probabilities $P_1, P_2, \ldots, P_N$, $\sum P_n = 1$ so, for example, with probability P_1 the signal is at frequency F_1.

A detector examines each of the N frequencies for a total of $T = T_1 + \ldots + T_N$ seconds. If t seconds is used to detect the signal **when it is present**, it is detected with probability

$$P_d = P_d(t) = 1 - \exp(-t/T)$$

so, the longer we search, the higher the chance of detecting the target.

1. (5 points) Write down the probability of detecting the target given an allocation $T_1, T_2, \ldots, T_N$
2. (5 points) What would be the probability of finding the signal if we only search in frequncy 1?
3. (5 points) What is the maximum probability of finding the signal if we only search in the one *mostly likely* frequency?

1. $P[\text{detect}] = \sum_{i=1}^{N} P_i \left(1 - \exp\left(-\frac{T_i}{T}\right)\right)$

2. $P[\text{find}] = P[\text{signal Locates in } F_1] \cdot P[\text{find} \mid \text{signal locates in } F_1]$

$= P_1 \left(1 - \exp\left(-\frac{T}{T}\right)\right) = P_1 (1 - e^{-1})$

$\approx P_1 \left(1 - \frac{1}{3}\right) = \frac{2}{3} P_1$

3. $P[\text{find}] = P[\text{signal Locates in } F_{max}] \cdot P[\text{find} \mid \text{signal locates in } F_{max}]$

$= P_{max}\left(1 - e^{-\frac{T}{T}}\right) = P_{max}(1 - e^{-1}) \approx P_{max}\left(1 - \frac{1}{3}\right) = \frac{2}{3} P_{max}$

here, $P_{max} = \max(P_1, P_2, \ldots, P_N)$

5 Multivariate RVs (20 pts)

Two statistically independent random variables **x** and **y** have mean values

$$E[\mathbf{x}] = 5 \quad (1)$$
$$E[\mathbf{y}] = -5 \quad (2)$$
$$E[\mathbf{x}^2] = 100 \quad (3)$$
$$E[\mathbf{y}^2] = 400 \quad (4)$$
$$E[(\mathbf{x}-5)(\mathbf{y}+5)] = 10 \quad (5)$$

Let

$$\mathbf{u} = \mathbf{x}+\mathbf{y} \quad (6)$$
$$\mathbf{v} = \mathbf{x}-\mathbf{y} \quad (7)$$

1. (5 pts) Find the mean value of **u**
2. (5 pts) Find the variance of **u**
3. (5 pts) Find the covariance of **u** and **v**
4. (5 pts) Find the minimum value over all a of $E[((\mathbf{u}) - a(\mathbf{v}))^2]$.

1. $E[u] = E[(x+y)] = E[x] + E[y] = 0$

2. $E[(x-5)(y+5)] = E[xy] - 5E[y] + 5E[x] - 25 = 10$

$\Rightarrow E[xy] = 10 + 25 + 5E[y] - 5E[x] = -15$

$Var[u] = E[u^2] - (E[u])^2 = E[u^2] = E[(x+y)^2]$

$= E[x^2] + 2E[xy] + E[y^2] = 100 + 400 - 30 = 470$

3. $Cov[u,v] = E[uv] - E[u]E[v] = E[(x+y)(x-y)] - (E[x]+E[y])(E[x]-E[y])$

$= E[x^2-y^2] - ((E[x])^2 - (E[y])^2)$

$= E[x^2] - E[y^2] - (E[x])^2 + (E[y])^2$

$= 100 - 400 - 25 + 25 = -300$

4. $E[((u)-a(v))^2] = E[(x+y-ax+ay)^2]$

$= E[((1-a)x+(1+a)y)^2] = (1-a)^2E[x^2] + (1+a)^2E[y^2] + 2(1-a^2)E[xy]$

$= 530a^2 + 600a + 470 \quad \Rightarrow$ when $a = -\frac{30}{53}$, $\min\{E[((u)-a(v))^2]\} \approx 300.1887$

6. **Max SNR (15pts)**

A signal

$$s_n = [+1 \;\; -1]$$

is received over an AWGN channel w_n resulting in

$$x_n = s_n + w_n$$

The average power in w_n is

$$P_w = E[w_n^2] = 100$$

The received signal x_n is processed using a correlator

$$V = \sum_{n=1}^{2} x_n c_n$$

1. (5pts) Determine the SNR in V for any c_n where

$$\text{SNR} = E(V^2|x = s)/E(V^2|x = w)$$

2. (5pts) Determine the best choice of c_n, to maximize the SNR.
3. (5pts) Give a numerical answer for the max SNR.

1. $SNR = \frac{(C_1S_1 + C_2S_2)^2}{(C_1^2 + C_2^2)E[W^2]}$

2. $\frac{dSNR}{dC_1} = 0 \Rightarrow S_1C_2 = C_1S_2$

$\frac{dSNR}{dC_2} = 0 \Rightarrow S_1C_2 = C_1S_2.$

$\Rightarrow C_n = CS_n$ (that is, $C_1 = CS_1$, $C_2 = CS_2$, and C is an arbitrary number except "0" here).

3. $SNR = \frac{(S_1^2 + S_2^2)^2}{(S_1^2 + S_2^2)E[W^2]} = \frac{2}{100} = \frac{1}{50}$

7 Continuous RVs (10 pts)

Two continuous independent RVs $\mathbf{x}, \mathbf{y}$ have an exponential pdfs with mean values

$$E[\mathbf{x}] = 2, \quad E[\mathbf{y}] = 4$$

1. (10 pts) Determine

$$P[\mathbf{y} > \mathbf{x}]$$

Because x, y have exponential pdfs,

and the mean for exponential distribution

is $E[x] = \frac{1}{\lambda_x}$, $E[y] = \frac{1}{\lambda_y}$,

We have $\lambda_x = \frac{1}{2}$, $\lambda_y = \frac{1}{4}$

So the pdfs for x, y are

$p(x) = \frac{1}{2} e^{-\frac{1}{2}x}$ $\quad$ $p(y) = \frac{1}{4} e^{-\frac{1}{4}y}$.

Because x, y are independent, the joint pdf for x & y is

$p(x,y) = p(x)\,p(y) = \frac{1}{8} e^{-\frac{1}{2}x} e^{-\frac{1}{4}y}$

$P[y>x] = \int_0^\infty \int_x^\infty p(x,y)\,dy\,dx = \int_0^\infty \int_x^\infty \frac{1}{8} e^{-\frac{1}{2}x} e^{-\frac{1}{4}y}\,dy\,dx$

$= \int_0^\infty \frac{1}{2} e^{-\frac{1}{2}x} \left(-e^{-\frac{1}{4}y}\right)\Big|_x^\infty dx = \int_0^\infty \frac{1}{2} e^{-\frac{1}{2}x} e^{-\frac{1}{4}x} dx$

$= \int_0^\infty \frac{1}{2} e^{-\frac{3}{4}x} dx = -\frac{1}{2} \cdot \frac{4}{3} \cdot e^{-\frac{3}{4}x}\Big|_0^\infty = \frac{2}{3}$