Hw7, EE416, Autumn 2009

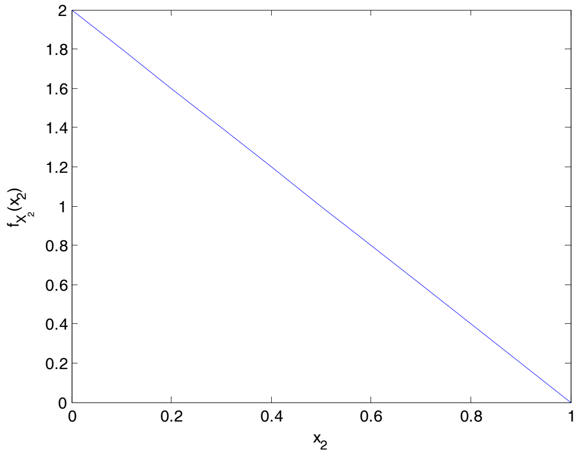
5．3

$$
\begin{aligned}
f_K[k] &= f_{K,\alpha}[k,\alpha=0.4]+f_{K,\alpha}[k,\alpha=0.6] \\
&= f_{K,\alpha}[k\,|\,\alpha=0.4]\mathrm{Pr}_\alpha[\alpha=0.4]+f_{K,\alpha}[k\,|\,\alpha=0.6]\mathrm{Pr}_\alpha[\alpha=0.6] \\
&= (0.4)(0.6)^{k-1}\times\frac{1}{2}+(0.6)(0.4)^{k-1}\times\frac{1}{2}, \qquad k\geq 1
\end{aligned}
$$

5．18

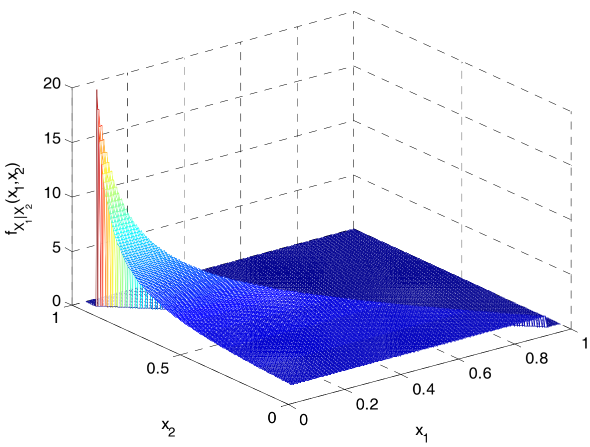
(a)

$$
f_{X_2}(x_2)=\int_0^{1-x_2} f_{X_1X_2}(x_1,x_2)\,\mathrm{d}x_1=\begin{cases}2(1-x_2), & 0\leq x_2\leq 1\\ 0, & \text{otherwise}\end{cases}
$$



(b)

$$
f_{X_1|X_2}(x_1\,|\,x_2)=\frac{f_{X_1X_2}(x_1,x_2)}{f_{X_2}(x_2)}=\begin{cases}\dfrac{1}{1-x_2}, & 0\leq x_1\leq 1-x_2,\text{ and } 0\leq x_2<1\\ 0, & \text{otherwise}\end{cases}
$$



(c)

$$
E[X_1\,|\,X_2]=\int_0^{1-x_2} x_1\times f_{X_1|X_2}(x_1\,|\,x_2)\,\mathrm{d}x_1=\int_0^{1-x_2} x_1\frac{1}{1-x_2}\,\mathrm{d}x_1=\frac{1-x_2}{2}, \quad 0\leq x_2\leq 1
$$

$$
E[X_1\,|\,X_2]=0, \quad \text{otherwise}
$$

6．4

(a) $\Pr\left[X \ge \frac{2}{\lambda}\right] \le \frac{E[X]}{2/\lambda} = \frac{1/\lambda}{2/\lambda} = \frac{1}{2}$

(b) $\Pr\left[X \ge \frac{2}{\lambda}\right] = \Pr\left[\left|X - \frac{1}{\lambda}\right| \ge \frac{1}{\lambda}\right] \le \frac{\sigma_x^2}{(1/\lambda)^2} = \frac{1/\lambda^2}{(1/\lambda)^2} = 1$

(c) $\Pr\left[\left(X - \frac{1}{\lambda}\right) \ge \frac{1}{\lambda}\right] \le \frac{\sigma_x^2}{\sigma_x^2 + (1/\lambda)^2} = \frac{1/\lambda^2}{1/\lambda^2 + (1/\lambda)^2} = \frac{1}{2}$

(d) $\Pr\left[X \ge \frac{2}{\lambda}\right] = \int_{2/\lambda}^{\infty} \lambda e^{-\lambda x}\, \mathrm{d}x = e^{-2} \approx 0.135$

6．7

(a) $E[Y_n] = E\left[\frac{1}{12}(X_{n-11} + \cdots + X_n)\right] = \frac{1}{12}\left(E[X_{n-11}] + \cdots + E[X_n]\right) = E[X_i] = 2$

$$\begin{aligned} Var[Y_n] &= Var\left[\frac{1}{12}(X_{n-11} + \cdots + X_n)\right] \\ &= \frac{1}{12^2} Var\left[\frac{1}{12}(X_{n-11} + \cdots + X_n)\right] \\ &= \frac{1}{12^2}\left(Var[X_{n-11}] + \cdots + Var[X_n]\right) \\ &= \frac{1}{12} Var[X_i] \\ &= \frac{3}{12} = \frac{1}{4} \end{aligned}$$

(b) The central limit theorem (CLT) states the mean of a sufficiently large number of independent random variables will be approximately Gaussian distributed. Assuming 12 is large enough, then Y_n is approximately a Gaussian r.v. with mean 2 and variance 1/4.

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sqrt{1/4}} e^{-\frac{(y-2)^2}{2(1/4)}} = \frac{2}{\sqrt{2\pi}} e^{-2(y-2)^2}$$

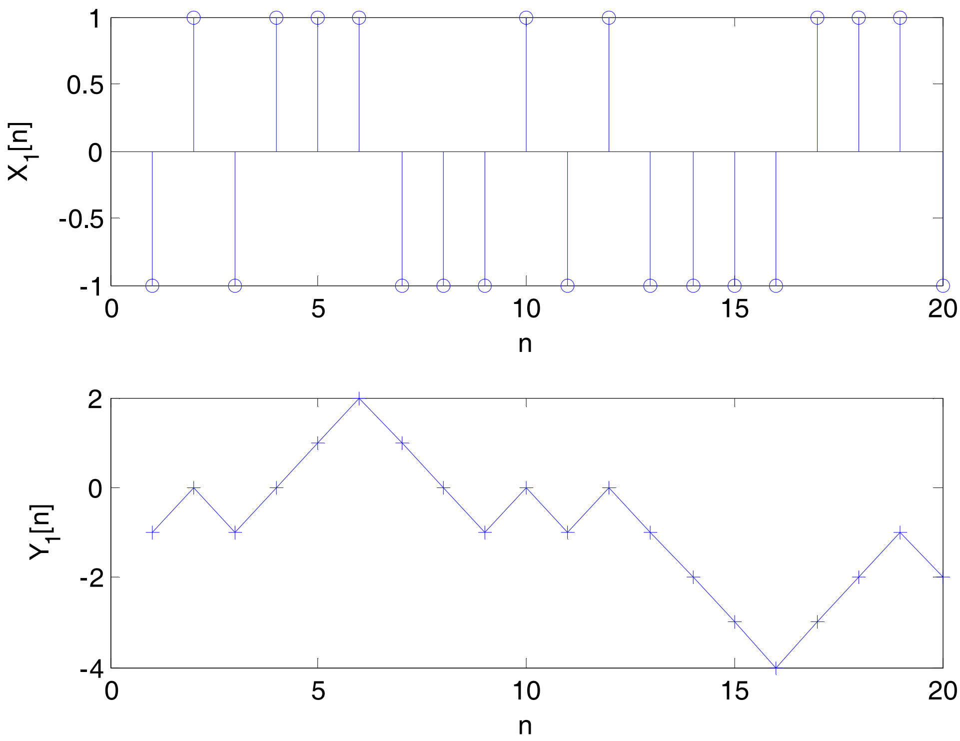
(c) $Var[Y_n] = \frac{1}{L} Var[X_i] = \frac{3}{L} \le 0.1$， then $L \ge 30$.

(d) For $L = 30$， $\sigma_Y = 0.1$. And $m_Y = 2$.

$$\begin{aligned} \Pr\left[|Y - m_Y| > 0.1\right] &= \Pr\left[|Y - 2| > 0.1\right] \\ &= 2\Pr[Y - 2 > 0.1] \\ &= 2Q\left(\frac{2.1 - m_Y}{\sigma_Y}\right) \\ &= 2Q\left(\frac{2.1 - 2}{\sqrt{0.1}}\right) \approx 0.7518 \end{aligned}$$

7. 5

(a,b) suppose $p=0.5$.



%matlab code

```
u=rand(20,1);
u(u<0.5)=-1;
u(u>0.5)=1;
subplot(2,1,1);
stem(u);
xlabel('n');
ylabel('X_1(n)')
y=cumsum(u);
subplot(2,1,2);
plot(y,'+-');
xlabel('n');
ylabel('Y_1(n)')
```

(c) $E\{Y[k]\}=E\{X[0]+\cdots+X[k]\}=E\{X[0]\}+\cdots+E\{X[k]\}=(k+1)E\{X[i]\}=(k+1)(2p-1)$

$Var\{Y[k]\}=Var\{X[0]+\cdots+X[k]\}=Var\{X[0]\}+\cdots+Var\{X[k]\}=(k+1)Var\{X[i]\}=(k+1)4p(1-p)$

both depend on time index k

(d) For $p=\frac{1}{2}$, $E\{Y[k]\}=0$, doesn't depend on k

$Var\{Y[k]\}=(k+1)$, depends on k

Grading

10 for 6.3,

20 for 5.118,

20 for 6.4,

25 for 6.7,

25 for 7.5.