Project 3.2

1(a)

H_1: X = A + N

H_0: X = N

Thus $\Pr\{x \mid H_0\}$ is a Gaussian distribution with mean 0 and variance σ^2; $\Pr\{x \mid H_1\}$ is a Gaussian distribution with mean A and variance σ^2, where σ^2 =2.5 given by SNR=20log10(A/σ).

The general rule to minimize the error probability, for any distribution, is that

$$\frac{\Pr\{x \mid H_0\}}{\Pr\{x \mid H_1\}} \underset{H_1}{\overset{H_0}{\gtrless}} \frac{P_1}{P_0}.$$

For both $\Pr\{x \mid H_0\}$ and $\Pr\{x \mid H_1\}$ being Gaussian distribution, the above general rule will reduce to a decision boundary that $x \underset{H_1}{\overset{H_0}{\gtrless}} \tau$.

$\tau = \frac{A}{2} + \frac{\sigma^2}{A} \ln \frac{P_0}{P_1}$ is given by $\frac{\Pr\{\tau \mid H_0\}}{\Pr\{\tau \mid H_1\}} = \frac{P_1}{P_0}$.

1(b)

ε_0 is the error probability when the acutal signal is H_0 but decoded as H_1; ε_1 is the error probability when the acutal signal is H_1 but decoded as H_0.

$$\varepsilon_0 = \Pr[x > \tau \mid H_0] = \int_\tau^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-0)^2}{2\sigma^2}} \mathrm{d}x = Q(\frac{\tau}{\sigma}),$$

$$\varepsilon_1 = \Pr[x < \tau \mid H_1] = \int_{-\infty}^\tau \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-A)^2}{2\sigma^2}} \mathrm{d}x = 1 - Q(\frac{\tau - A}{\sigma}),$$

or $\varepsilon_1 = Q(\frac{A-\tau}{\sigma})$, because $Q(x) + Q(-x) = 1$.

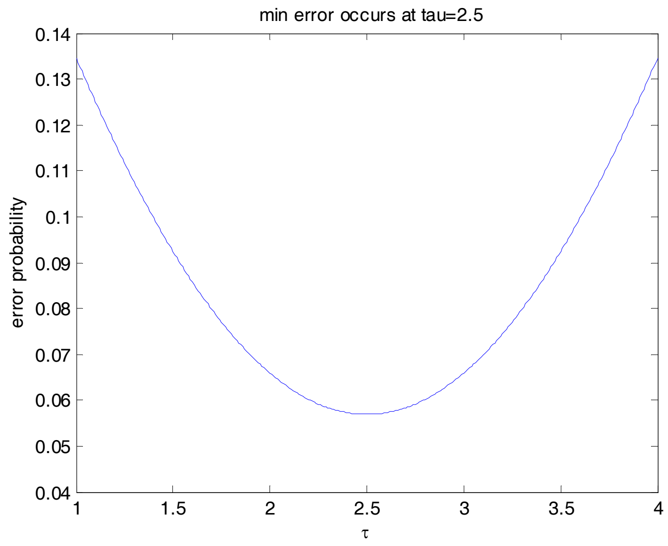
$\Pr[\text{error}] = \varepsilon_0 P_0 + \varepsilon_1 P_1$.

% the matlab code for (b)

```
Q = inline('1/2*erfc(x/sqrt(2))','x'); % define the Q function
A = 5; sigma = sqrt(2.5); P0 = 1/2; P1 = 1/2;
tau = 1:0.01:4;
```

```
error0 = Q( tau./sigma );
error1 = Q( (A-tau)./sigma );
error=error0*P0+error1*P1;

plot(tau,error)
xlabel('\tau');
ylabel('error probability');

%find the min error probablity
[val,index]=min(error);
title(['min error occurs at tau=',num2str(tau(index))]);
```

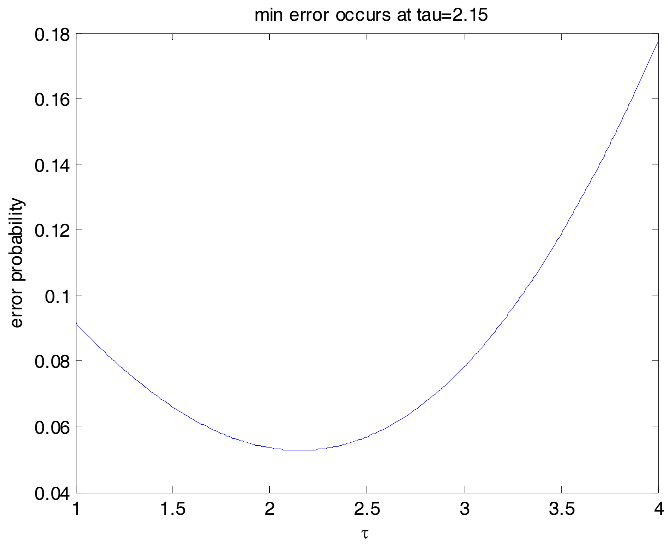
% matlab output



1(c)

The min error should occur at $\tau = \frac{A}{2} + \frac{\sigma^2}{A}\ln\frac{P_0}{P_1} \approx 2.15$.

% the matlab code is the same as (b) except for different values of P_0 and P_1.
% matlab output



3(a) i.

% this code generate 1000 H_0 samples and 1000 H_1 sample. And classify the samples by the rule that if x<tau then decoded as H_0; if x>tau then decoded as H_1.

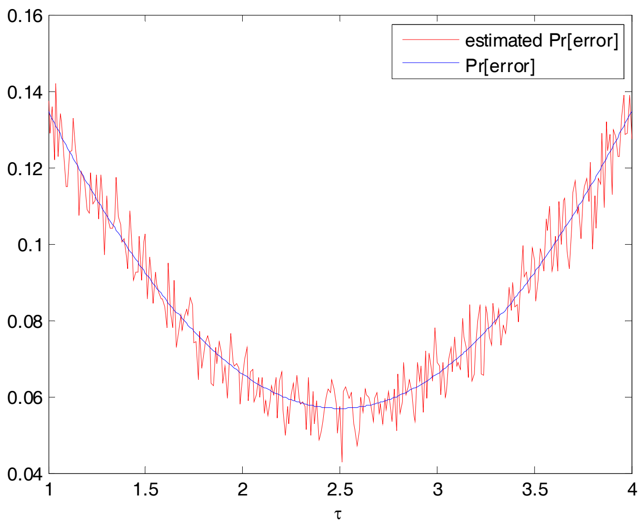
```
Q = inline('1/2*erfc(x/sqrt(2))','x'); % define the Q function
A = 5; sigma = sqrt(2.5);  P0 = 0.5; P1 = 0.5;% Initialize parameters

% results of problem 1
tau = 1:0.01:4;
error0 = Q( tau./sigma );
error1 = Q( (A-tau)./sigma );
error = error0*P0+error1*P1;
pD = 1-error1;
pFA = error0;

error_all=zeros(0,0);
pD_all=zeros(0,0);
pFA_all=zeros(0,0);
for tau = 1:0.01:4
    x0 = 0 + sigma * randn(1000, 1); % Generate Gaussian samples
    x1 = A + sigma * randn(1000, 1);
    % Determine error_i for Hi
    error0 = length(find(x0 > tau))/1000;
    error1 = length(find(x1 <= tau))/1000;
    error_all=[error_all error0*P0+error1*P1];
    pD_all=[pD_all 1-error1];
    pFA_all=[pFA_all error0];
end
```

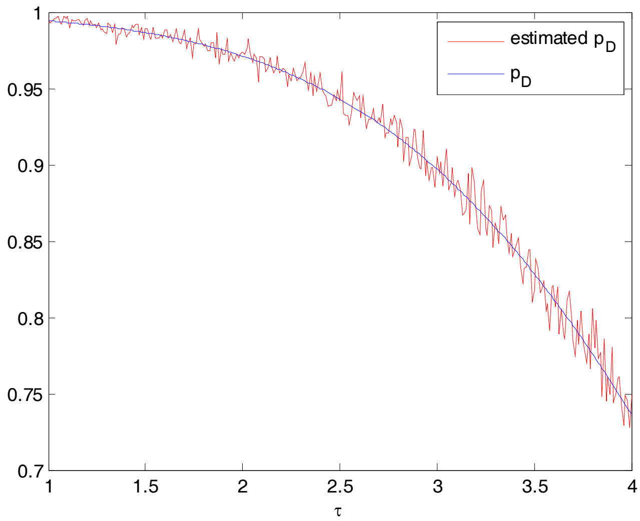
```
plot(1:0.01:4,error_all,'r-',1:0.01:4,error);
xlabel('\tau');
legend('estimated Pr[error]','Pr[error]');
figure;
plot(1:0.01:4,pD_all,'r-',1:0.01:4,pD);
xlabel('\tau');
legend('estimated p_{D}','p_{D}');
figure;
plot(1:0.01:4,pFA_all,'r-',1:0.01:4,pFA);
xlabel('\tau');
legend('estimated p_{FA}','p_{FA}');
```

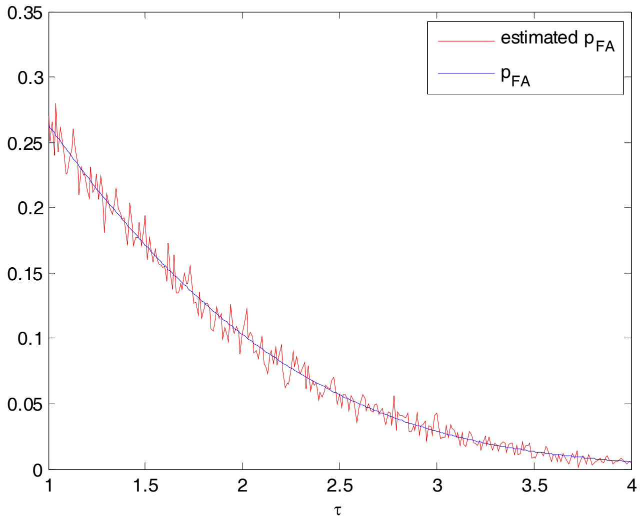
3(a) ii.

% matlab output

% compare the theoretic Pr[error], P_D, P_{FA} and the estimated ones







At $\tau = 2.5$,

$\hat{\varepsilon}_0 = 0.04$, $\hat{\varepsilon}_1 = 0.058$.	$\varepsilon_0 = 0.0569$, $\varepsilon_1 = 0.0569$.
Detection probability $\hat{p}_D = 1 - \hat{\varepsilon}_1 = 0.9420$	Detection probability $p_D = 1 - \varepsilon_1 = 0.9431$
False alarm probability $\hat{p}_{\mathrm{FA}} = \hat{\varepsilon}_0 = 0.04$	False alarm probability $p_{\mathrm{FA}} = \varepsilon_0 = 0.0569$

4.5

(a) $E\{X\}=\int_{-b}^{b} x\frac{1}{2b}\,\mathrm{d}x=0$

(b) $f_{X|X>0}(x\,|\,X>0)=\frac{1}{b}, 0<x\le b$

$$E\{X\,|\,X>0\}=\int_{0}^{b} x\frac{1}{b}\,\mathrm{d}x=\frac{b}{2}$$

4.6

(a)

$$\Pr\{I=-1\,|\,I>0\}=\frac{\Pr\{\{I=-1\}\cap\{I>0\}\}}{\Pr\{I>0\}}=0$$

$$\Pr\{I=0\,|\,I>0\}=\frac{\Pr\{\{I=0\}\cap\{I>0\}\}}{\Pr\{I>0\}}=0$$

$$\Pr\{I=1\,|\,I>0\}=\frac{\Pr\{\{I=1\}\cap\{I>0\}\}}{\Pr\{I>0\}}=\frac{0.375}{0.375+0.0625+0.375}\approx 0.4615$$

$$\Pr\{I=2\,|\,I>0\}=\frac{\Pr\{\{I=2\}\cap\{I>0\}\}}{\Pr\{I>0\}}=\frac{0.0625}{0.375+0.0625+0.375}\approx 0.0769$$

$$\Pr\{I=3\,|\,I>0\}=\frac{\Pr\{\{I=3\}\cap\{I>0\}\}}{\Pr\{I>0\}}=\frac{0.375}{0.375+0.0625+0.375}\approx 0.4615$$

$$E\{I\,|\,I>0\}=\sum_{i=-1}^{3} i\cdot\Pr\{I=i\,|\,I>0\}=2$$

(b)

$$\Pr\{I=-1\,|\,I\le 0\}=\frac{0.125}{0.125+0.0625}=0.6667$$

$$\Pr\{I=0\,|\,I\le 0\}=\frac{0.0625}{0.125+0.0625}=0.3333$$

$$\Pr\{I=1\,|\,I\le 0\}=0$$

$$\Pr\{I=2\,|\,I\le 0\}=0$$

$$\Pr\{I=3\,|\,I\le 0\}=0$$

$$E\{I\,|\,I\le 0\}=-1\times\Pr\{I=-1\,|\,I\le 0\}+0\times\Pr\{I=0\,|\,I\le 0\}=-0.6667$$

5.9

(a)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X_1X_2}(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2 = 1$$

$$\int_0^3\int_0^1 C\left(3-x_1x_2\right)\mathrm{d}x_1\mathrm{d}x_2$$

$$=\int_0^3 C\left(3x_1-\frac{x^2{}_1}{2}x_2\right)\Bigg|_{x_1=0}^{1}\mathrm{d}x_2$$

$$=\int_0^3 C\left(3-\frac{1}{2}x_2\right)\mathrm{d}x_2$$

$$=C\left(3x_2-\frac{1}{2}\frac{x^2{}_2}{2}\right)\Bigg|_{x_2=0}^{3}$$

$$=C\frac{27}{4}=1$$

Thus $C=\dfrac{4}{27}$.

(b)

$$f_{X_1}(x_1)=\int_{-\infty}^{\infty} f_{X_1X_2}(x_1,x_2)\mathrm{d}x_2$$

$$=\int_0^3\frac{4}{27}\left(3-x_1x_2\right)\mathrm{d}x_2=\frac{4}{27}\left(3x_2-x_1\frac{x^2{}_2}{2}\right)\Bigg|_{x_2=0}^{3}=\frac{4}{27}\left(9-x_1\frac{9}{2}\right), 0\le x_1\le 1$$

$$f_{X_2}(x_2)=\int_{-\infty}^{\infty} f_{X_1X_2}(x_1,x_2)\mathrm{d}x_1$$

$$=\int_0^1\frac{4}{27}\left(3-x_1x_2\right)\mathrm{d}x_1=\frac{4}{27}\left(3x_1-\frac{x^2{}_1}{2}x_2\right)\Bigg|_{x_1=0}^{1}=\frac{4}{27}\left(3-x_1\frac{1}{2}\right), 0\le x_2\le 3$$

(c)

$F_{X_1X_2}(x_1,x_2)=0$, $\quad x_1<0$ or $x_2<0$

$F_{X_1X_2}(x_1,x_2)=\int_0^{x_2}\int_0^{x_1}\frac{4}{27}\left(3-u_1u_2\right)\mathrm{d}u_1\mathrm{d}u_2=\frac{4}{27}\left(3x_1x_2-\frac{x^2{}_1x^2{}_2}{4}\right)$, $\quad 0\le x_1\le 1$ and $0\le x_2\le 3$;

$F_{X_1X_2}(x_1,x_2)=\int_0^{x_2}\int_0^{1}\frac{4}{27}\left(3-u_1u_2\right)\mathrm{d}u_1\mathrm{d}u_2=\frac{4}{27}\left(3x_2-\frac{x^2{}_2}{4}\right)$, $\quad x_1>1$ and $0\le x_2\le 3$;

$F_{X_1X_2}(x_1,x_2)=\int_0^{3}\int_0^{x_1}\frac{4}{27}\left(3-u_1u_2\right)\mathrm{d}u_1\mathrm{d}u_2=\frac{4}{27}\left(9x_1-\frac{9x^2{}_1}{4}\right)$, $\quad 0\le x_1\le 1$ and $x_2>3$;

$F_{X_1X_2}(x_1,x_2)=1$, $\quad x_1>1$ and $x_2>3$

5.25

X,Y~exp(λ), where $\lambda = 1$.

Method 1:

(a)

Pr{message received within 1/4 hour]

=1-Pr{X>1/4 hour, Y>1/4 hour}

=1-Pr{X>1/4 hour}Pr{Y>1/4 hour}

=1-$e^{-1/4}$ $e^{-1/4}$

=1-$e^{-1/2}$

(b)

Pr{message received and verified within 1/4 hour}

=Pr{X<1/4 hour, Y<1/4 hour}

= Pr{X<1/4 hour}Pr{Y<1/4 hour}

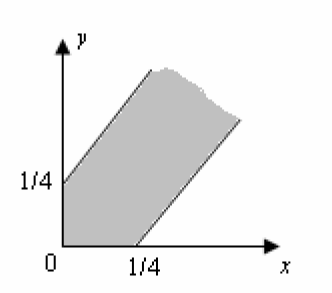
=(1-$e^{-1/4}$) (1-$e^{-1/4}$)

(c)

i. If X<Y, then the receiving time is X, the verification time is Y, and {verification-receiving<=1/4} is {Y-X<=1/4};

ii. If Y<X, then the receiving time is Y, the verification time is X, and {verification-receiving<=1/4} is {X-Y<=1/4};

Thus the event is {|X-Y|<1/4}.



Pr{message verified within 1/4 hour after received| Y>X }

= Pr{|X-Y|<1/4}

$$=2\int_0^{\infty}\int_x^{x+1/4} e^{-x}e^{-y}\,dy\,dx$$

=1-$e^{-1/4}$

The integral area is show on the right.

(d) Pr{verified within 1/4 hour| received within 1/4 hour}

$$= \frac{\text{Pr\{message received and verified within 1/4 hour\}}}{\text{Pr\{received within 1/4 hour\}}}$$

$$= \frac{\text{Pr\{ verified within 1/4 hour\}}}{\text{Pr\{received within 1/4 hour\}}}$$

$$=\frac{(1-e^{-1/4})(1-e^{-1/4})}{1-e^{-1/2}}$$

Some has the following incorrect answer. I'll explain and give the correct way.

The incorrect answer. Some hw argued that given that either X or Y <= 1/4, which means at least one r.v. <=1/4, without loss of generality we can assume that X <=1/4. Then the probability will reduce to

Pr{verified within 1/4 hour| received within 1/4 hour}

=Pr{X<=1/4, Y<=1/4| X<=1/4}

= Pr{Y<=1/4}.

The correct way assuming X<=1/4 is that

Pr{verified within 1/4 hour| received within 1/4 hour}

Pr{verified within 1/4 hour| received within 1/4 hour, X<=1/4}* Pr{ X<=1/4| received within 1/4 hour}+

Pr{verified within 1/4 hour| received within 1/4 hour, X>1/4}* Pr{ X>1/4| received within 1/4 hour}

= Pr{verified within 1/4 hour| X<=1/4}* Pr{ X<=1/4| received within 1/4 hour}+0

= Pr{Y<=1/4 | X<=1/4}* Pr{ X<=1/4| received within 1/4 hour}+0

$= \Pr\{Y<=1/4\} * \frac{\Pr\{X<=1/4\}}{\Pr\{\text{ received within 1/4 hour}\}}$

$= (1-e^{-1/4}) * \frac{(1-e^{-1/4})}{1-e^{-1/2}}$

Method 2: Using property of exponential r.v.

The properties can be found at http://en.wikipedia.org/wiki/Exponential_distribution

(a)

Pr{message received within 1/4 hour]

=Pr{min(X,Y)<1/4 hour}

$=1-e^{-1/2}$,

where Z=min(X,Y) is an exponential random variable with rate $\lambda_Z = \lambda_X + \lambda_Y = 2$.

(c)

Two cases: either X>Y or X<Y.

If Y>X, Pr{message verified within 1/4 hour after received| Y>X }

=Pr{Y-X<1/4 hour| Y>X }

=1-Pr{Y-X>1/4 hour| Y>X }

=1-Pr{Y > X +1/4 hour| Y>X}

=1-Pr{Y > 1/4 hour} (here the memoryless property of exponential r.v.)

$=1-e^{-1/4}$

If X>Y, Pr{message verified within 1/4 hour after received| X>Y }

$=1-e^{-1/4}$

Thus Pr{message verified within 1/4 hour after received]

= Pr{message verified within 1/4 hour after received| Y>X } Pr{Y>X }+

Pr{message verified within 1/4 hour after received| X>Y } Pr{ X>Y }

$=(1-e^{-1/4})*1/2+(1-e^{-1/4})*1/2$

$=1-e^{-1/4}$

Grading:

30 for problem 1 in project 3.2, where 10 for each a) b) c);

10 for problem 3 a) in project 3.2;

10 for 4.5;

10 for 4.6;

20 for 5.9, where 5 for a), 5 for b) and 10 for c);

20 for 5.25.