EE416 Random Signals for Communications and Signal Processing

Solutions for Homework 1, Fall 2009

1. Let A, B be finite sets. Which is smaller in cardinality: $A \cap B$ or $A \cup B$? When are they equal?

Solution:

$A \cap B$ is usually smaller than $A \cup B$, as shown in Fig 1.1. And $A \cap B$ equals to $A \cup B$ when A equals to B.

It is because $A \bigcup B = (A \bigcap B) \bigcup (A^c \bigcap B) \bigcup (A \bigcap B^c)$

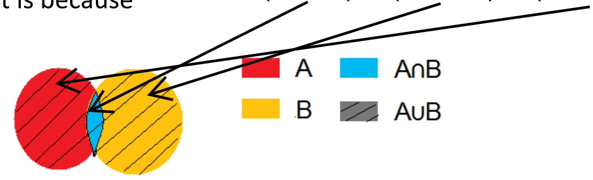

Fig 1.1

|A| denotes the cardinality of A, then we have

$$\begin{cases} |A \bigcap B| < |A \bigcup B| & A \neq B \\ |A \bigcap B| = |A \bigcup B| & A = B \end{cases}$$

2. If P(A)=a, P(B)=b, then $P(A|B) \geq \frac{a+b-1}{b}$, prove using basic properties.

Solution:

$$P(A|B) = \frac{P(A \bigcap B)}{P(B)}$$

$$= \frac{P(A) + P(B) - P(A \bigcup B)}{P(B)}$$

$$= \frac{a + b - P(A \bigcup B)}{b}$$

$$\because P(A \bigcup B) \leq 1$$

$$\therefore P(A|B) = \frac{a + b - P(A \bigcup B)}{b} \geq \frac{a+b-1}{b}$$

Proved.

3. $\because P(A\bigcup B) = P(A) + P(B) - P(A\bigcap B)$

and

$P(A\bigcap B) \geq 0$

$\therefore P(A\bigcup B) \leq P(A) + P(B)$

And when A and B mutually exclusive, that is, $P(A\bigcap B) = 0$, we have

$$P(A\bigcup B) = P(A) + P(B)$$

It is similar for the case of three sets.

$$\begin{aligned}
&\because P(A_1 \cup A_2 \cup A_3) \\
&= P(A_1) + P(A_2) + P(A_3) - P(A_1A_2) - P(A_2A_3) - P(A_1A_3) + P(A_1A_2A_3) \\
&and \\
&P(A_1A_2) + P(A_2A_3) + P(A_1A_3) - P(A_1A_2A_3) \\
&= P(A_1A_2) + P(A_2A_3) + P(A_1A_3) - P((A_1A_2)\bigcap(A_2A_3)) \\
&\geq P(A_1A_3) \geq 0 \\
&\therefore P(A_1 \cup A_2 \cup A_3) \leq P(A_1) + P(A_2) + P(A_3)
\end{aligned}$$

This can be extended to the case of N, where

$$P(A_1 \cup A_2 \cup \cdots \cup A_N) = \sum_{j=1}^{N} P(A_j) - \sum_{j<k} P(A_jA_k) + \cdots + (-1)^{(N-1)} P(A_1A_2 \cdots A_N)$$

(this can be proved with induction method)

And

$$P(A_1 \cup A_2 \cup \cdots A_N) \leq \sum_{k=1}^{N} P(A_k)$$

Proved

Solution2:

By using

$$P(A\bigcup B) \leq P(A) + P(B)$$

We have

$$\begin{aligned}
&P(A_1 \cup A_2 \cup A_3) = P((A_1 \cup A_2) \cup A_3) \\
&\leq P(A_1 \cup A_2) + P(A_3) \\
&\leq P(A_1) + P(A_2) + P(A_3)
\end{aligned}$$

This can be extended for N sets and thus we have

$$
\begin{aligned}
&P(A_1 \cup A_2 \cup \cdots \cup A_N) \\
&= P((A_1 \cup A_2 \cup \cdots \cup A_{N-1}) \cup A_N) \\
&\le P(A_1 \cup A_2 \cup \cdots \cup A_{N-1}) + P(A_N) \\
&\le \cdots \\
&\le P(A_1) + P(A_2) + \cdots + P(A_N) \\
&= \sum_{k=1}^{N} P(A_k)
\end{aligned}
$$

Thus "union bound" proved

4. Geometric Probability: throw coins in grid.

Solution:

As all the grids are the same, we'll look at a single grid first. It is like Fig 1.2

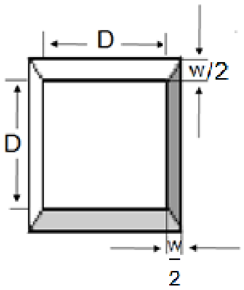

Fig 1.2

To find out the probability of winning, we need to divide this grid into to area, that is,

P(whole grid)=P(winning area)+P(not winning area)

And as the coin will definitely fall onto one of these grid areas, we have

P(whole grid)=1

And as we can see from Fig 1.2,

$$Area_{whole\_grid} = (D + \frac{W}{2} + \frac{W}{2})^2 = (D + W)^2$$

A coin fall into the grid only if the center of the coin (circle) fall at least R distance from the lines. This is shown in Fig 1.3.

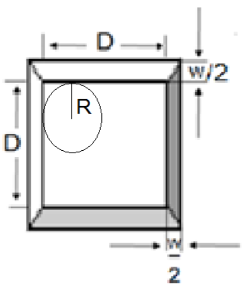

Fig 1.3

So the winning area is a square in the grid with distance larger than R from each of the four lines.

This is shown in Fig 1.4.

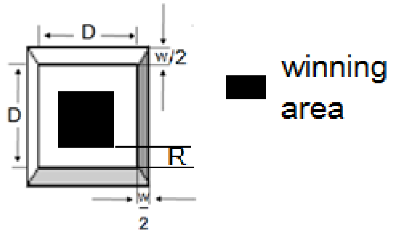

Fig 1.4

As it is defined the probability of distribution is uniform, we have

$$P(winning) = \frac{P(winning\_area)}{P(whole\_grid)}$$

$$= \frac{(D-2R)^2}{(D+W)^2}$$

To find out D for P(winning)=0.5, we simply let

$$\frac{(D-2R)^2}{(D+W)^2} = 0.5$$

This two degree equation has two roots. We will abort the root that is smaller than the diameter of the coin, because we know when the grid is smaller than the coin, a coin could never fall into a grid without touching any lines.
With W=5mm,
We have the following result:

Coin	Diameter (mm)	D(mm)
dime	17.9	73.19
cent	19.0	76.94
nickel	21.2	84.45
quarter	24.3	95.04
Half	30.6	116.55
Dollar	38.1	142.15

5. Geometric Probability: if two signals are jamming.
Solution:
Let t_1 represent the arrival time of one signal, and t_2 represent the arrival time of the other signal. The area where signals are jammed is shown in Fig 1.5.

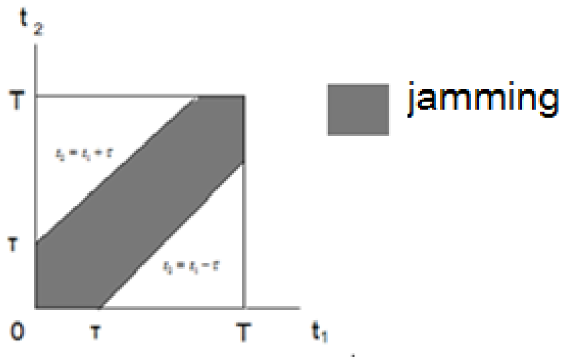

Fig 1.5

i) Thus, the probability of jamming is

$$P_J = \frac{jam\_area}{whole\_area} = \frac{T^2 - (T-\tau)^2}{T^2}$$

ii) The probability of both signal clearly received is

$$P_C = 1 - P_J = \frac{(T-\tau)^2}{T^2}$$

iii) When $\tau << T$,

$$P_J = 2\frac{\tau}{T} - (\frac{\tau}{T})^2 \approx 2\frac{\tau}{T} = 2\gamma$$

iv) When $T \to \infty$, we have $\gamma \to 0$. Thus, $P_J \to 0$

6. A communication link network.

Solution:

We know that P(success on 1 trial)=0.5

N links are used when each link is independent. So we have

1st case: P(system succeeds)=P(at least 1 link works)=1-P(no links work)=1-P(the 1st link fails)...P(the n-th link fails)
=1-0.5^n

$$1 - 0.5^n \geq 0.999$$

$$0.001 \geq 0.5^n$$

$$\lg(0.001) \geq n \lg 0.5$$

$$n \geq 10$$

So smallest n=10

2nd case:

P(system succeeds)=P(at least 2 links work)=1-P(no links work)-P(only 1 link works)
=1-P(the 1st link fails)...P(the n-th link fails)-
P(n choose 1)P(the chosen link works)P(the remaining n-1 links fail)
=1-0.5^n-n(0.5)^n

Using matlab or other tools, we have n>=14.

Thus, the smallest n is 14. It does not double.

7. Networks

Solution:

(1) O=(at least one of A's fails) ∪ (all of B's fails)   or   O=1- (all A work)*(at least one of B works)

(2) $P_O = (1 - p_k{}^N) + (1 - q_k)^M - (1 - p_k{}^N)(1 - q_k)^M$

(3) With p_k=0.99, q_k=0.5,

We have $P_O = (1 - 0.99^N) + 0.5^M - (1 - 0.99^N)0.5^M$

A sample matlab program for N=1

```
for m=1:50
po=(1-0.99^1)+0.5^m-(1-0.99^1)*0.5^m;
if po<=0.1
break;
end
end
```

the m needed is as follows

N	M	N	M
1	4	6	5
2	4	7	5
3	4	8	6
4	4	9	7
5	5	10	8

(4) if N>10, P_O can never be lesser than 0.1. Because for N>10, P(the serial part is out)>0.1, the probability that the whole system is out could not be lesser than 0.1, no matter how large M is, according to the results in (1).

Grading rules:

the full score is 100.

the full score for each question is

1: 10 credit (5 for each task)

2: 15 credit

3: 15 credit (5 for each task)

4: 15 credit (2 for sketch, 10 for theoritical result, 3 for 6 calculations)

5: 15 credit (7, 2, 3, 3 for each task, respectively)

6: 15 credit (5 for each task)

7: 15 credit (5, 5, 3, 2 for each task, respectively