mean = 60
st dev = 23
high score = 100

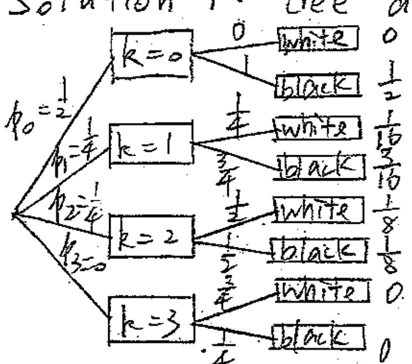
Problem 1

(10 points) A discrete RV **k** takes on values in the sample space $S = \{0, 1, \ldots, 3\}$. The probability of each outcome is given by $p_k = P(\mathbf{k} = k)$, where

$$\begin{aligned} p_0 &= 1/2 && (1)\\ p_1 &= 1/4 && (2)\\ p_2 &= 1/4 && (3)\\ p_3 &= 0 && (4) \end{aligned}$$

and otherwise is zero. Let **k** represent a number of **white balls**. Take these and add $4 - \mathbf{k}$ **black balls** and place into bottle. Draw one at random. Determine the probability that this ball is white. Show all your work.

Solution 1: tree diagram



so $P[\text{white}]$
$= 0 + \frac{1}{16} + \frac{1}{8} + 0$
$= \frac{3}{16}$

Solution 2: Total Probability

$P[\text{white}] = P[\text{white}|p_0]\,p_0 + P[\text{white}|p_1]\,p_1 + P[\text{white}|p_2]\,p_2 + P[\text{white}|p_3]\,p_3$
$= 0 \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} + \frac{3}{4} \cdot 0$
$= \frac{3}{16}$

$P[\text{white}|p_k]$ is the probability that a ball is white when there are k white balls and 4-k black balls, and therefore is $\frac{k}{4}$.

Problem 2

(15 points) A Bernoulli generator puts out an unlimited binary sequence, $x_1, x_2, \ldots$ Assume that

$$\begin{aligned} P(1) &= p = 0.6 && (5)\\ P(0) &= q = 0.4 && (6) \end{aligned}$$

(5 points) Find the probability that the first $x = 1$ occurs on draw k, for any $k = 1, 2, 3, \ldots$

(5 points) Find the probability that the first *match* occurs on draw k, for any $k = 1, 2, 3, \ldots$. A match is either a 1 or a 0. For example, in 011 the match occurs on draw $k = 3$.

(5 points) Find the mean time to a match.

Show all your work.

Over

PROBLEM 2

(a) $P(\text{Time to first } 1) = P_k = p(1-p)^{k-1} \quad k = 1, 2, 3, \ldots$

geometric

(b) I had intended "match" to mean "match the first draw".

Eg, $k = 2$: xx ~~ef~~ for $x \in \{0,1\}$

$k = 3$ xyx, xyy for $x, y \in \{0,1\}$ $x \neq y$

The $P(k=2) = q^2 + p^2$

$P(k=3) = 2qp^2 + 2q^2p$

(c) with mean value

$$2(q^2+p^2) + 3 \cdot 2 \cdot (qp^2 + pq^2)$$

I also gave credit to solid attempts at find $P(\text{match on draw } k)$ when a match was 2 consecutive values.

Problem 3

(15 points) One Bernoulli generator puts out an unlimited binary sequence, $x_1, x_2, \ldots$ with $P(1) = 1/2$. A second independent Bernoulli generator also puts out an unlimited binary sequence, $y_1, y_2, \ldots$ also with $P(1) = 1/2$.

Let other binary sequences be formed according to

$$z_k = \max(x_k, y_k) \quad w_k = \min(x_k, y_k)$$

and total of $n = 10$ samples are recorded. Let

$$X = \sum_{k=1}^{n} x_k$$

$$Z = \sum_{k=1}^{n} z_k = \sum_{k=1}^{n} \max(x_k, y_k)$$

$$W = \sum_{k=1}^{n} z_k = \sum_{k=1}^{n} \min(x_k, y_k)$$

1) (5points) What is the probabability that the $X \leq 5$
2) (5points) What is the probabability that the $W \leq 5$
3) (5points) What is the probabability that the $Z \leq 5$
Explain your work.

1) Because x_k is either 0 or 1, $X = \sum_{k=1}^{n} x_k$ is the number of times when $x_k = 1$. It is therefore a binomial distribution.

$$P(X \leq 5) = \sum_{k=0}^{5} P(X=k) = \sum_{k=0}^{5} \binom{10}{k} \left(\frac{1}{2}\right)^k \left(1-\frac{1}{2}\right)^{(10-k)} = \sum_{k=0}^{5} \binom{10}{k} \left(\frac{1}{2}\right)^{10}$$

This is because $P(X=k)$ is the probability where there are k out of 10 x's equal to 1; while other $(10-k)$ x's equal to 0.

2) $$w_k = \min(x_k, y_k) = \begin{cases} x_k & x_k \leq y_k \\ y_k & x_k > y_k \end{cases}$$

as x_k, y_k can only be 0 or 1,

$$w_k = \begin{cases} 0 & \text{when } x_k = 0, \text{ or } y_k = 0 \\ 1 & \text{when: } x_k = 1, \text{ and } y_k = 1 \end{cases}$$

Because x_k and y_k are independent

So the probability $P(W_k=0) = P(x_k=0 \text{ or } y_k=0) = \frac{3}{4}$

$P(W_k=1) = P(x_k=1 \text{ and } y_k=1) = P(x_k=1)\,P(y_k=1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

From the analysis in (1), we can see W is also a binomial distribution.

So $P(W \le 5) = \sum_{k=0}^{5} P(W=k) = \sum_{k=0}^{5} \binom{10}{k} \left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{(10-k)}$

3) Similarly,

because $Z_k = \max(x_k, y_k) = \begin{cases} x_k & x_k \ge y_k \\ y_k & x_k < y_k \end{cases}$

So $Z_k = \begin{cases} 0 & x_k=0 \text{ AND } y_k=0 \\ 1 & x_k=1 \text{ OR } y_k=1 \end{cases}$

$P(Z_k=0) = P(x_k=0, y_k=0) = P(x_k=0)P(y_k=0) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$P(Z_k=1) = 1 - P(Z_k=0) = \frac{3}{4}$

$P(Z \le 5) = \sum_{k=0}^{5} P(Z=k) = \sum_{k=0}^{5} \binom{10}{k} \left(\frac{3}{4}\right)^k \left(\frac{1}{4}\right)^{(10-k)}$.

Problem 4

(15 points) A continuous RV x has probability density function (pdf) $p(x)$

$$p(x) = Ax^2, \ \ 0 < x < 1$$

and

$$p(x) = -Bx \quad -1 < x < 0$$

otherwise it is zero.

1) (5 pts) Determine conditions on A, B so this is a pdf.

2) (5 pts) Find A, B so that $P(\mathrm{x} > 0) = 1/2$.

3) (5 pts) Find $E\{\mathrm{x}\}$, the expected value of x.

Integrate carefully and show your work.

1) pdf has two important properties:

① pdf is non-negative at all points.

$p(x) = Ax^2 \geq 0, \ 0 \leq x < 1$

$\Rightarrow \underline{A \geq 0}$

$p(x) = -Bx \geq 0, \ -1 < x < 0$

$\Rightarrow \underline{B \geq 0}$

② $\int_{-\infty}^{\infty} \text{pdf}\, dx = 1$

$\int_{-\infty}^{\infty} p(x)dx = \int_0^1 Ax^2 dx + \int_{-1}^{0} -Bx\, dx = 1$

So $\frac{B}{2} + \frac{A}{3} = 1$

2) $P(x > 0) = \int_0^\infty p(x) dx$

$= \int_0^1 Ax^2 dx = \frac{1}{2}$

So $A = \frac{3}{2}$

Because we have $\frac{B}{2} + \frac{A}{3} = 1$ from (1)

So $B = 1$

3) $E\{x\} = \int_{-\infty}^{\infty} x\, p(x)\, dx = \int_0^1 Ax^3 dx + \int_{-1}^{0} -Bx^2 dx$

$= -\frac{B}{3} + \frac{A}{4}$

plug in the data from (2),

$E\{x\} = -\frac{B}{3} + \frac{A}{4} = \frac{1}{24}$

Problem 5

(15points) An exponential RV x has unit mean, $E\{\mathrm{x}\} = \mu > 0$, with pdf

$$p(x) = \frac{1}{\mu} \exp(-x/\mu), \ x \geq 0$$

1) (10pts) Find the median, the point z for which

$$P(\mathrm{x} > z) = P(\mathrm{x} \leq z)$$

2) (5 pts) Assume that x represents the available signal-to-noise (SNR) on a communications link. The bit error rate (BER) of the system drops as BER $= \frac{1}{2}\exp(-\mathrm{x})$. **Determine** $E[\mathrm{BER}]$. Show your work

1) Because $P(x > z) = P(x \leq z)$

So $\int_{-\infty}^{z} p(x) dx = \int_z^\infty p(x)\, dx$

$\int_{-\infty}^{z} \frac{1}{\mu} \exp(-\frac{x}{\mu})\, dx = \int_z^\infty \frac{1}{\mu} \exp(-\frac{x}{\mu})\, dx$

$z = \mu \ln 2$

2) $E\{BER\} = E\{\frac{1}{2} e^{-x}\}$

$= \int_{-\infty}^{\infty} \frac{1}{2} e^{-x} p(x) dx$

$= \int_0^\infty \frac{1}{2} e^{-x} \frac{1}{\mu} e^{-\frac{x}{\mu}} dx$

$= \frac{1}{2(\mu+1)} = [2(\mu+1)]^{-1}$

Problem 6

(15 points) Consider a positive random variable $\mathbf{t} > 0$. You know that

$$P[\mathbf{t} > t] = \exp(-at^3), \quad 0 < t, \ a > 0$$

1. (10 pts) Determine the pdf $p(t)$ of RV $\mathbf{t}$
2. (5 pts) Determine, in terms of a, the probability that

$$\max(\mathbf{t}_1, \mathbf{t}_2) > 1$$

assuming that $\mathbf{t}_1$ and $\mathbf{t}_2$ are IID.
Hint: replace the event on the max using a condition on both $\mathbf{t}_1$ and $\mathbf{t}_2$. Recall that we know how to handle unions, intersections, and complements.

1) a CDF is defined as $P[\underline{t} \le t]$, so the CDF is $1 - P(\underline{t} > t)$
So $CDF = 1 - e^{-at^3}$, $t>0$, $a>0$
pdf is the derivative of CDF,
so pdf $p(t) = \frac{d}{dt} CDF = 3at^2 e^{-at^3}$

2) $P[\max(t_1, t_2) > 1] = 1 - P[\max(t_1, t_2) \le 1] = 1 - P[t_1 \le 1, t_2 \le 1]$
$= 1 - P[t_1 \le 1] P[t_2 \le 1]$ (this is because t_1 and t_2 are independent)
$= 1 - (1 - e^{-a})(1 - e^{-a}) = 1 - (1 - 2e^{-a} + e^{-2a}) = 2e^{-a} - e^{-2a}$
↑ this is the CDF when $t=1$, according to the definition of CDF

Problem 7

(15 points) A continuous RV $\mathbf{x}$ has an exponential probability density function (pdf) $f(x)$ with mean one;

$$f(x) = \exp(-x), x > 0$$

and zero elsewhere.
Consider the transformation

$$y = g(x) = e^x$$

Write down the resulting $p(y)$, the pdf of y and sketch. Show where this pdf is non-zero.

According to the formula in Chapter 3.6 in the textbook,

$$p(y) = \frac{1}{|dy/dx|} f(x) \Big|_{x=g^{-1}(y)} = \frac{1}{|e^x|} e^{-x} \Big|_{x=\ln y} = e^{-2\ln y} = \frac{1}{y^2}$$

Because $x>0$,
so $y = e^x > 1$