CSS 455 Winter 2012 C. Jackels

Activity No. 3 and 3b (Answer Key) January 12, 2012

The purpose of this activity is to explore formation of the Jacobian matrix and to use an iterative procedure to solve the catenary equation.

1. **The Jacobian** matrix arises in the solution of a system of equations in multiple unknowns. In Turner's example (p.45), he defines the functions to be solved as:

$$f_1(x,y) = 4x^2 + y^2 - 4 = 0$$

$$f_2(x,y) = x^2 y^3 - 1 = 0$$

To solve the system, you need the Jacobian, which consists of the full set of partial derivatives of both functions

The solution is:

$$\mathbf{J} = \begin{pmatrix} 8x & 2y \\ 2xy^3 & 3x^2y^2 \end{pmatrix} = \begin{pmatrix} f_{1x} & f_{1y} \\ f_{2x} & f_{2y} \end{pmatrix} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$

With you partner, verify that understand how each of these partial derivatives were obtained and that you understand all of the notation used above. *If you have any questions at all, have them answered now!*

In MP1, the following equations are presented:

$$\lambda \cosh\left(\frac{L_1}{\lambda}\right) = \lambda + s$$

$$\lambda \cosh\left(\frac{L - L_1}{\lambda}\right) = \lambda + s + H$$

Write them below in the form $f_1(\lambda, L_1) = 0$ *and* $f_2(\lambda, L_1) = 0$. *This is just a rearrangement exercise to get them in the form f(x)=0.*

$$f_1(\lambda, L_1) = \lambda \cosh\left(\frac{L_1}{\lambda}\right) - \lambda - s = 0$$

$$f_2(\lambda, L_1) = \lambda \cosh\left(\frac{L - L_1}{\lambda}\right) - \lambda - s - H = 0$$

See reverse side

Find the four partial derivatives of these two functions $\partial f_1/\partial \lambda$ $\partial f_1/\partial L_1$ $\partial f_2/\partial \lambda$ $\partial f_2/\partial L_1$

$$\partial f_1(\lambda, L_1)/\partial \lambda = \cosh(L_1/\lambda) - \left(\frac{L_1}{\lambda}\right)\sinh(L_1/\lambda) - 1 = \mathbf{J}_{11}$$

$$\partial f_1(\lambda, L_1)/\partial L_1 = \sinh(L_1/\lambda) = \mathbf{J}_{12}$$

$$\partial f_2(\lambda, L_1)/\partial \lambda = \cosh\left(\tfrac{L-L_1}{\lambda}\right) - \left(\tfrac{L-L_1}{\lambda}\right)\sinh\left(\tfrac{L-L_1}{\lambda}\right) - 1 = \mathbf{J}_{21}$$

$$\partial f_2(\lambda, L_1)/\partial L_1 = -\sinh\left(\tfrac{L-L_1}{\lambda}\right) = \mathbf{J}_{22}$$

If λ plays the role of x_1 and L1 that of x_2, indentify the derivatives as J_{11}, J_{12}, J_{21}, and J_{22}.

See above. If you are having trouble getting these partial derivatives right, be sure to see me or a classmate to work them out. Remember, treat the other variable as a constant when differentiating with respect to one of the two variables. L is a constant, of course.

2. **Iterative solution to an equation.** (This is Act 3b, Part I) The purpose of this exercise is to exhibit the fixed point iteration method of solution. Consider the catenary equation :

$$\lambda \cosh\left(L/2\lambda\right) = \lambda + s$$

Write this equation in the form *f(λ) = 0,* with the constants *s=40 and L=300.:*

$$f(\lambda) = \lambda \cosh\left(\frac{L}{2\lambda}\right) - \lambda - s = \lambda \cosh\left(\frac{150}{\lambda}\right) - \lambda - 40 = 0$$

Rearrange the equation to provide at least *two different* equations of the form *λ= g(λ)*

$$\lambda = g(\lambda) = \lambda \cosh\left(\frac{150}{\lambda}\right) - 40$$

$$\lambda = g(\lambda) = \frac{40}{\cosh(150/\lambda) - 1}$$

$$\lambda = g(\lambda) = \frac{40 + \lambda}{\cosh(150/\lambda)}$$

$$\lambda = g(\lambda) = \frac{150}{\cosh^{-1}\left(\frac{40+\lambda}{\lambda}\right)}$$

See reverse side

Selecting one of them, use the initial guess λ_0 =270 and evaluate the right hand side. This result is the next guess at the solution λ_1 . Now plug this new value into the right hand side to obtain the subsequent guess λ_3. Continue until it is clear whether the value is converging to the correct value of 287.68.

- **a.** Using the first equation above, the iteration procedure yields: 270, 272.7494, 275.0462, 276.9723, and on toward convergence.
- **b.** Using the second equation above: 270, 252.6350, 220.3805, 166.1697, 91.7737,… not toward convergence, except perhaps at lambda =0, which is a unphysical root and not interesting.
- **c.** Using the third equation above: 270, 267.6264, 264.9153, 261.8128, 258.2550, 254.1644, 249.4476, …It is moving more slowly than case (b) above, but is going to the same place – toward unphysical solution at lambda =0.
- **d.** Using the fourth equation above: 270, 278.9; 283.35; 285.56… It appears to be moving toward convergence.

Overall message is that you can create a number of algebraically correct iterants for many equations. The convergence properties differ greatly, however. You need to test such an iterant in the range of application before using it. Note, we have only tested at one starting point. The three equations (and others) may behave differently at others. One can analyze these equations mathematically to predict this behavior. For our purposes, however, it is sufficient to test the iterant in the range of application. Often, such an experiment is much quicker and easier than doing the math analysis.

Activity 3b, Part II, has no answers – it was an exercise in using Matlab and parallel computing.

See reverse side