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CSS 455 Winter 2012 C. Jackels
Activity No. 12 February 24, 2012
Names (must be present):

Part I. Numerical Quadrature.

The general formula we will be working with in this exercise is given below. The idea will be to arrive at three approximations for an integral and cast each of them into this general form:

$$\int_a^b f(x)dx \approx \int_a^b p(x)dx = (b-a)\sum_{k=0}^{N-1} c_k f(x_k)$$

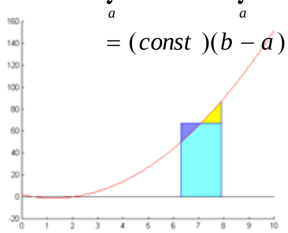
In each case, we will work with a geometrical approximation to the area beneath the curve over the domain (a,b).

1. **Rectangular or mid-point approximation.** This uses a zero-order polynomial (N=1) to interpolate:

Zero order polynomial: Assume the area is equal to that of the rectangle formed by the line at the midpoint of the region.

$$\int_a^b f(x)dx \approx \int_a^b p(x)dx = \int_a^b (const)\,dx =$$

$$= (const)(b-a) = f\left(\tfrac{a+b}{2}\right)\cdot(b-a)$$



Interpolation is of zero order.

$$\int_a^b f(x)dx \approx \int_a^b p(x)dx = (b-a)\sum_{k=0}^{N-1} c_k f(x_k)$$

Comparing the general quadrature formula above to the specific formula for this approximation:

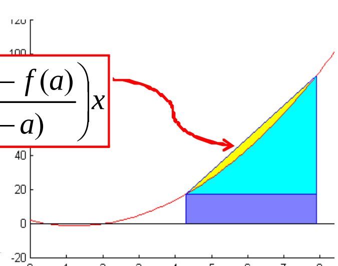
What is the value of the weight c_k ?

What is the value of x_k ?

2. **Trapezoidal (first order, N=2)** Approximation. The formula for the area comes from geometry or from integrating the expression for *p(x)*.

Trapezoid Approx: Assume the area is equal to that of the trapezoid formed by the line between the limits.

$$p(x) = f(a) + \left(\frac{f(b)-f(a)}{(b-a)}\right)x$$



$$\int_a^b f(x)dx \approx \int_a^b p(x)dx =$$

$$= f(a)\cdot(b-a) + \tfrac{1}{2}(b-a)\cdot[f(b)-f(a)]$$

Convince yourself that the formula beneath the graph (for area) is correct.

Compare the general formula at right to the specific formula for this case (N=2).

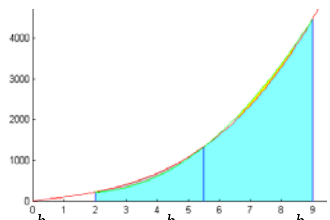
$$\int_a^b f(x)dx \approx \int_a^b p(x)dx = (b-a)\sum_{k=0}^{N-1} c_k f(x_k)$$

What are the values of the weights c_k ?

What is the values of the x_k 's used to evaluate $f(x_k)$?

3. **Quadratic (second order, N=3)** Approximation, also known as Simpson's Rule.

Simpson's Rule: Assume the area is formed by a parabola interpolating the limits and midpoint.



$$\int_a^b f(x)dx \approx \int_a^b p(x)dx = \int_a^b \left[\alpha + \beta x + \gamma x^2\right]dx$$

As in the previous (linear) case, the parameters of the interpolant (here α, β, and γ) are expressed in terms of the coordinates $a, \frac{a+b}{2}, b$ and the value of the function at those points $f(a), f(\frac{a+b}{2}), f(b)$

When these values are substituted in *p(x)* and the integration is carried out, the area beneath the curve is:

$$\int_a^b f(x)dx \approx \tfrac{1}{6}(b-a)\cdot[f(a) + 4f\left(\tfrac{a+b}{2}\right) + f(b)]$$

Compare the general formula above to the specific formula for this case (N=3).

$$\int_a^b f(x)dx \approx \int_a^b p(x)dx = (b-a)\sum_{k=0}^{N-1} c_k f(x_k)$$

What are the values of the weights c_k ?

What is the values of the x_k 's used to evaluate $f(x_k)$?

4. **Cubic Case (N=4).** In practice, you obtain the weights from a table, evaluate the function at the appropriate values of x and directly compute the estimate for the integral. For the N=4 approximate, we are told that the weights for the cubic interpolating polynomial for the above problem are given as c = [1/8, 3/8, 3/8, 1/8]. As in all of these examples, the x-values are evenly distributed over the domain (a,b). Write a formula analogous to the one for the N=3 interpolant at right, but for the case with N=4 (a cubic interpolant):

$$\int_a^b f(x)dx \approx \tfrac{1}{6}(b-a)\cdot[f(a) + 4f\left(\tfrac{a+b}{2}\right) + f(b)]$$