Astr 511: Galactic Astronomy

Winter Quarter 2015, University of Washington, Željko Ivezić

Review Lecture:

Stellar kinematics: a bit of theory

Outline

1. **Introduction to potentials and orbits**
   - **Spherical potentials**
   - **Axial potentials**
   - **Stellar orbits**
   - **Epicycle approximation**

2. **Statistical analysis: basic equations**
   - **Boltzmann equation**
   - **Jeans' equations**

3. **Applications of Jeans' aquations**

- **Asymmetric drift for disk stars**
- Local mass density
- The shape of local velocity ellipsoid
- Halo mass density profile

4. The Scattering of Disk Stars

For the last four, see Lecture 9.

Reading:

- Binney & Merrifield: ch. 10
- Smith et al. 2009 (MNRAS 399, 1223): Sec. 3.2 at least
- Carollo et al. 2010 (ApJ 712, 692): Sec. 8 at least

Spherical Systems – Important Quantities

The velocity of a test particle on a circular orbit is the **circular speed**, v_c. Setting the centripetal acceleration equal to the force we get

$$v_c^2 = r\frac{\mathrm{d}\Phi}{\mathrm{d}r} = r|\mathbf{F}| = \frac{GM(r)}{r}. \tag{1}$$

The circular speed is a measure of the mass interior to r, $M(r)$.

If v_c as a function of r is known, and we *assume that the potential is spherical*, we can compute the mass as a function of r (not the case for a non-spherical distribution.)

Another important quantity is the **escape speed**, v_e, defined by

$$v_e(r) = \sqrt{2|\Phi(r)|}. \tag{2}$$

This definition comes from setting the kinetic energy of a star equal to the absolute value of its potential energy. That is, stars with positive total energy are not bound to the system. In order for a star to escape from from the gravitational field represented by Φ, it is necessary that its speed be greater than v_e. This can be used to get the local Φ of the galaxy.

Spherical Systems – Simple Examples

Point mass:

$$\Phi(r) = -\frac{GM}{r} \quad ; \quad v_c(r) = \sqrt{\frac{GM}{r}} \quad ; \quad v_e(r) = \sqrt{\frac{2GM}{r}}. \tag{3}$$

Whenever the circular speed declines as $r^{-1/2}$, it is referred to as **Keplerian**. It usually implies that there is no significant mass at that radius.

Homogeneous sphere:

$$M = \frac{4}{3}\pi r^3 \rho \quad ; \quad v_c = \sqrt{\frac{4\pi G\rho}{3}} r. \tag{4}$$

The equation of motion for a particle in such a body is

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{GM(r)}{r^2} = -\frac{4\pi G\rho}{3} r, \tag{5}$$

which describes a harmonic oscillator with period

$$T = \sqrt{\frac{3\pi}{G\rho}}. \tag{6}$$

Spherical Systems – Simple Examples

Independent of r, if a particle is started at r, it will reach the center in a time

$$t_{dyn} = \frac{T}{4} = \sqrt{\frac{3\pi}{16G\rho}}, \tag{7}$$

known as the **dynamical time**. Although this result is only true for a homogeneous sphere, it is a common practice to use this definition with any system of density ρ.

By integrating the density for a homogeneous sphere, we can get the potental:

$$\Phi = \begin{cases} -2\pi G\rho(a^2 - \frac{1}{3}r^2), & r < a \\ -\frac{4\pi G\rho a^3}{3r}, & r > a. \end{cases}$$

One would expect the center of a galaxy to have a potential of this type if there is no cusp in the central density (implying a linear rise in v_c).

Spherical Systems – Simple Examples

Isochrone potential:

$$\Phi(r) = -\frac{GM}{b + \sqrt{b^2 + r^2}}. \tag{8}$$

This has the nice property of going from a harmonic oscillator in the middle to a Keplerian potential at large r, with the transition occurring at a scale b.

The circular speed is

$$v_c^2 = \frac{GMr^2}{(b+a)^2 a}, \tag{9}$$

where $a \equiv \sqrt{b^2 + r^2}$.

Using Poisson's equation ($\nabla^2\Phi = 4\pi G\rho$), we can find the density:

$$\rho(r) = \frac{1}{4\pi G r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = M\left[\frac{3(b+a)a^2 - r^2(b+3a)}{4\pi(b+a)^3 a^3}\right]. \tag{10}$$

So the central density is

$$\rho(0) = \frac{3M}{16\pi b^3}, \tag{11}$$

and the asymptotic density is

$$\rho(r) \approx \frac{bM}{2\pi r^4}. \tag{12}$$

Other commonly discussed profiles are **modified Hubble profile** and **power-law profile**.

Potential–Density Pairs

Simple models can be used to illustrate the dynamics of axisymetric galaxies.

Plummer's (1911) model: spherically symmetric

Kuzmin's (1956) model: infinitely thin disk (aka *Toomre's model 1*)

Plummer–Kuzmin models': introduced by Miyamoto & Nagai (1975), smooth transition from Plummer's to Kuzmin's models

Logarithmic potentials: the circular speed is a constant at large radii

The Milky Way Potential

The most popular Milky Way models are double exponential disk (thin and thick in the Z direction, also exponential dependence in the R direction), with a power-law or logarithmic halo.

In general, the potentials are constrained using the spatial distribution of stars, or the kinematic information (e.g. the rotation curve; later in this lecture).

Some recent good reviews:

Bahcall (1986, ARA&A 24, 577)

Gilmore, Wyse & Kuijken (1989, ARA&A 27, 555)

Majewski (1993, ARA&A 31, 575)

Freeman & Bland-Hawthorn (2002, ARA&A 40, 487)

Orbits in Static Spherical Potentials

The problem: given the initial conditions $\mathbf{x}(t_o)$ and $\dot{\mathbf{x}}(t_o)$, and the potential $\Phi(r)$, find $\mathbf{x}(t)$.

Orbits in spherical potentials are easy to consider and lead to some important concepts.

- Some general considerations
- Example 1: Spherical harmonic oscillator: $\Phi(r) = A + B\,r^2$
- Example 2: Point mass potential: $\Phi(r) = \frac{-GM}{r}$
- Example 3: Isochrone potential: $\Phi(r) = \frac{-GM}{b+\sqrt{b^2+r^2}}$

General considerations

The initial conditions are 6-dimensional and thus a general solution includes six orbital parameters. (aka *constants of motion*)

The equation of motion in a spherical potential is:

$$\ddot{\mathbf{r}} = F(r)\hat{\mathbf{e}}_r, \tag{13}$$

i.e. the force is always radial!

Crossing through by $\mathbf{r}$, we show that the angular momentum vector, $\mathbf{L} \equiv \mathbf{r} \times \dot{\mathbf{r}}$ is conserved:

$$\frac{\mathrm{d}}{\mathrm{dt}}(\mathbf{r} \times \dot{\mathbf{r}}) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{dt}} \times \frac{\mathrm{d}\mathbf{r}}{\mathrm{dt}} + \mathbf{r} \times \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{dt}^2} = F(r) \times \hat{\mathbf{e}}_r = 0 \tag{14}$$

Therefore, the motion is constrained to the plane perpendicular to $\mathbf{L}$, and can be fully described in cylindrical coordinate system, r and ψ ($\mathbf{v} = \dot{r}\,\hat{\mathbf{e}}_r + r\dot{\psi}\,\hat{\mathbf{e}}_\psi$)

General considerations

The equations of motion in the plane are

$$\begin{aligned} \ddot{r} - r\dot{\psi}^2 &= F(r) \\ 2\dot{r}\dot{\psi} + r\ddot{\psi} &= 0. \end{aligned}$$

The second equation comes from $r^2\dot{\psi} = L =$ const. (note that this is the second Kepler's law!)

$\dot{\psi}$ can be eliminated using $\dot{\psi} = L/r^2$, leading to a one-dimensional equation of motion:

$$\ddot{r} - L^2/r^3 = F(r). \tag{15}$$

This equation motivates a definition of an effective potential

$$-\nabla\Phi_{\text{eff}} \equiv F(r) + L^2/r^3, \tag{16}$$

and thus

$$\Phi_{\text{eff}}(r) \equiv \Phi(r) + \frac{L^2}{2r^2}. \tag{17}$$

General considerations

The energy per unit mass is

$$E = \frac{1}{2}v^2 + \Phi(r) = \frac{1}{2}(\dot{r}^2 + r^2\dot{\psi}^2) + \Phi(r) = \frac{1}{2}\left(\dot{r}^2 + \Phi_{\text{eff}}(r)\right). \quad (18)$$

For bound orbits r oscillates between an inner radius, or pericenter (r_{min}), and an outer radius, or apocenter (r_{max}). The radial period is

$$T_r = 2\int_{r_{\text{min}}}^{r_{\text{max}}} \left(\sqrt{2[E - \Phi_{\text{eff}}(r)]}\right)^{-1} dr \quad (19)$$

The pericenter and apocenter are the solutions of $\Phi_{\text{eff}}(r) = E$.

General considerations

The azimuthal period is

$$T_\psi = \frac{2\pi}{\Delta\psi} T_r \tag{20}$$

where

$$\Delta\psi = 2L \int_{r_{\mathrm{min}}}^{r_{\mathrm{max}}} \left(r^2 \sqrt{2[E - \Phi_{\mathrm{eff}}(r)]}\right)^{-1} dr \tag{21}$$

The orbit is closed only for $\Delta\psi = k(2\pi)$ – in general case, the orbit forms a rosette.

The orbital precession rate:

$$\Omega_p = \frac{\Delta\psi - 2\pi}{T_r} \tag{22}$$

General considerations

If we eliminate t rather than ψ, then we have an equation for the orbit's shape. In terms of the variable $u \equiv 1/r$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\psi^2} + u = -\frac{F(u)}{L^2 u^2} \quad \Rightarrow \quad \frac{\mathrm{d}^2 u}{\mathrm{d}\psi^2} = \zeta(u). \tag{23}$$

This is a second order differential equation for $u(\psi)$, where $\zeta(u)$ and the initial conditions are presumably specified.

Let's now look at specific examples.

The harmonic potential

$$\Phi = \Phi_0 + \frac{1}{2}\Omega^2 r^2. \tag{24}$$

Generated by homogeneous density distribution.

The motion decouples in cartesian co-ordinates to $\ddot{x} = -\Omega^2 x$ and $\ddot{y} = -\Omega y$, and the solution is:

$$x = X\cos(\Omega t + \phi_x), \quad y = Y\sin(\Omega t + \phi_y), \tag{25}$$

where X, Y, ϕ_x and ϕ_y are arbitrary constants (determined from initial conditions).

This is the equation for an ellipse **centered** on the origin.

Orbits are closed since the periods for x and y oscillations are identical.

Point mass (Keplerian) potential

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\psi^2} + u = \frac{GM}{L^2} \quad \Rightarrow \quad u = \frac{GM}{L^2}[1 + e\cos(\psi - \psi_0)]. \tag{26}$$

This is the equation for an ellipse with one **focus at the origin** and eccentricity e (the first Kepler's law). The semi-major axis is $a = L^2/GM(1 - e^2)$.

The motion is periodic in ψ with period 2π. This gives a closed orbit with

$$T_r = T_\psi = 2\pi\sqrt{\frac{a^3}{GM}} = 2\pi GM(2|E|)^{-3/2} \tag{27}$$

Note that $T^2 \propto a^3$ – the third Kepler's law!

Isochrone Potential

$$\Phi(r) = \frac{-GM}{b + \sqrt{b^2 + r^2}} \tag{28}$$

More extended than point mass, less extended than harmonic potential.

T_r same as for the Keplerian case ($T_r = 2\pi GM(2|E|)^{-3/2}$).

However,

$$\Delta\psi = \pi \left[1 + \frac{L}{\sqrt{L^2 + 4GMb}} \right] \tag{29}$$

i.e. $\pi < \Delta\psi < 2\pi$, and hence **the orbits are not closed!**

Axisymmetric Potentials

The problem: given the initial conditions $\mathbf{x}(t_o)$ and $\dot{\mathbf{x}}(t_o)$, and the potential $\Phi(R, z)$, find $\mathbf{x}(t)$.

A better description of real galaxies than spherical potentials, and the orbital structure is much more interesting.

- Poisson's equation for axisymmetric potentials, meridional plane

- Non-axisymmetric examples

- Epicycle approximation

Axisymmetric Potentials

The equations of motion in an axisymmetric potential (cylindrical coordinates) are

$$\ddot{R} = -\frac{\partial \Phi_{\mathrm{eff}}}{\partial R} \tag{30}$$

and

$$\ddot{z} = -\frac{\partial \Phi_{\mathrm{eff}}}{\partial z} \tag{31}$$

where

$$\Phi_{\mathrm{eff}} \equiv \Phi + \frac{L_z^2}{2R^2} \tag{32}$$

Also

$$\frac{\mathrm{d}}{\mathrm{d}t}(R^2\dot{\phi}) = 0 \quad \Rightarrow \quad R^2\dot{\phi} = L_z. \tag{33}$$

Axisymmetric Potentials

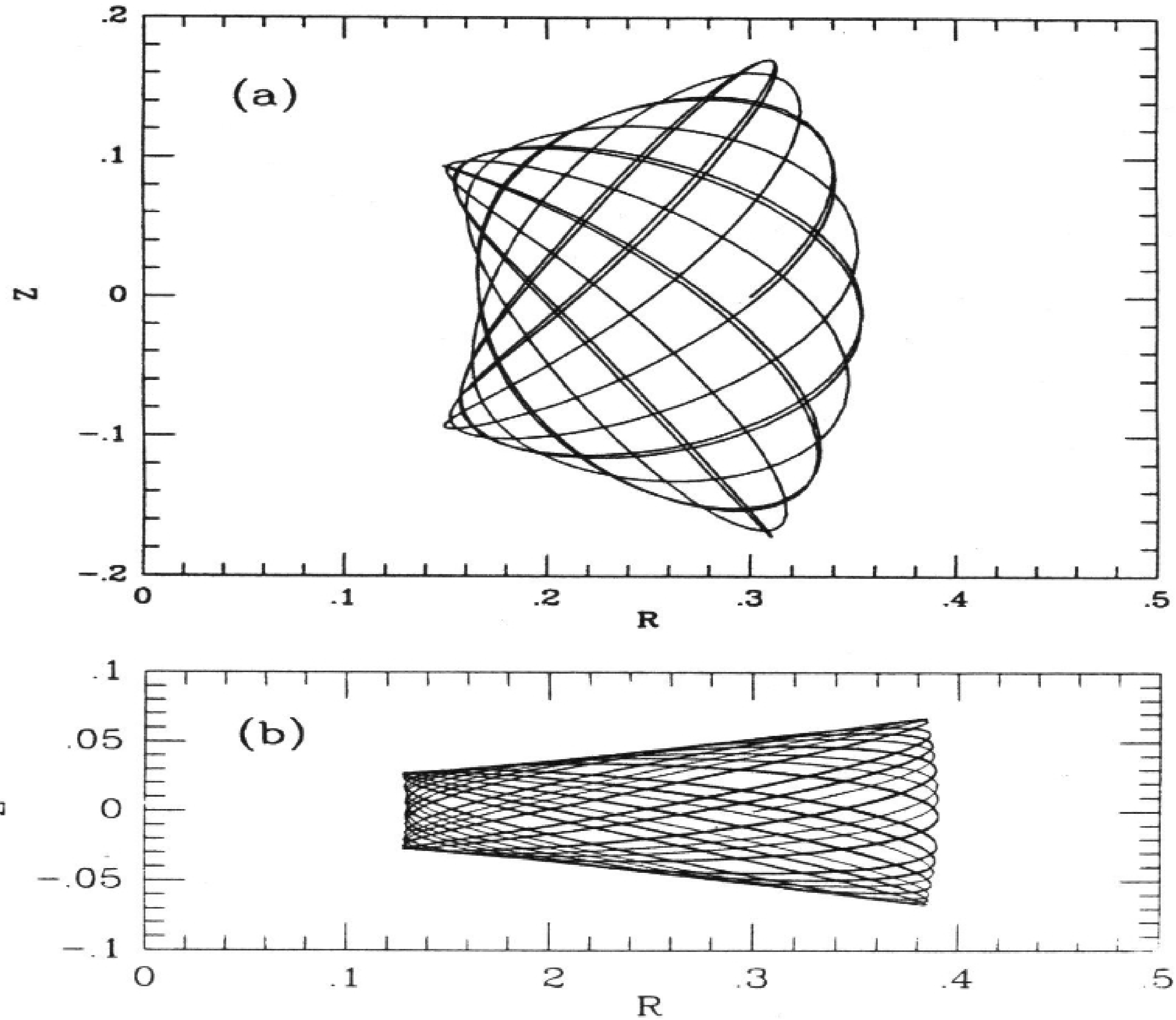
Hence, if we solve the first two equations, the solution for ϕ can be obtained from the third equation as

$$\phi(t) = \phi(t_o) + \dot{\phi}(t_o)\, R^2(t_o) \int_{t_o}^{t} dt'/R^2(t') \tag{34}$$

Meridional plane: non-uniformly rotating plane. The three-dimensional motion in the cylindrical (R, z, ϕ) space is reduced to a two-dimensional problem in **Cartesian** coordinates R and z.

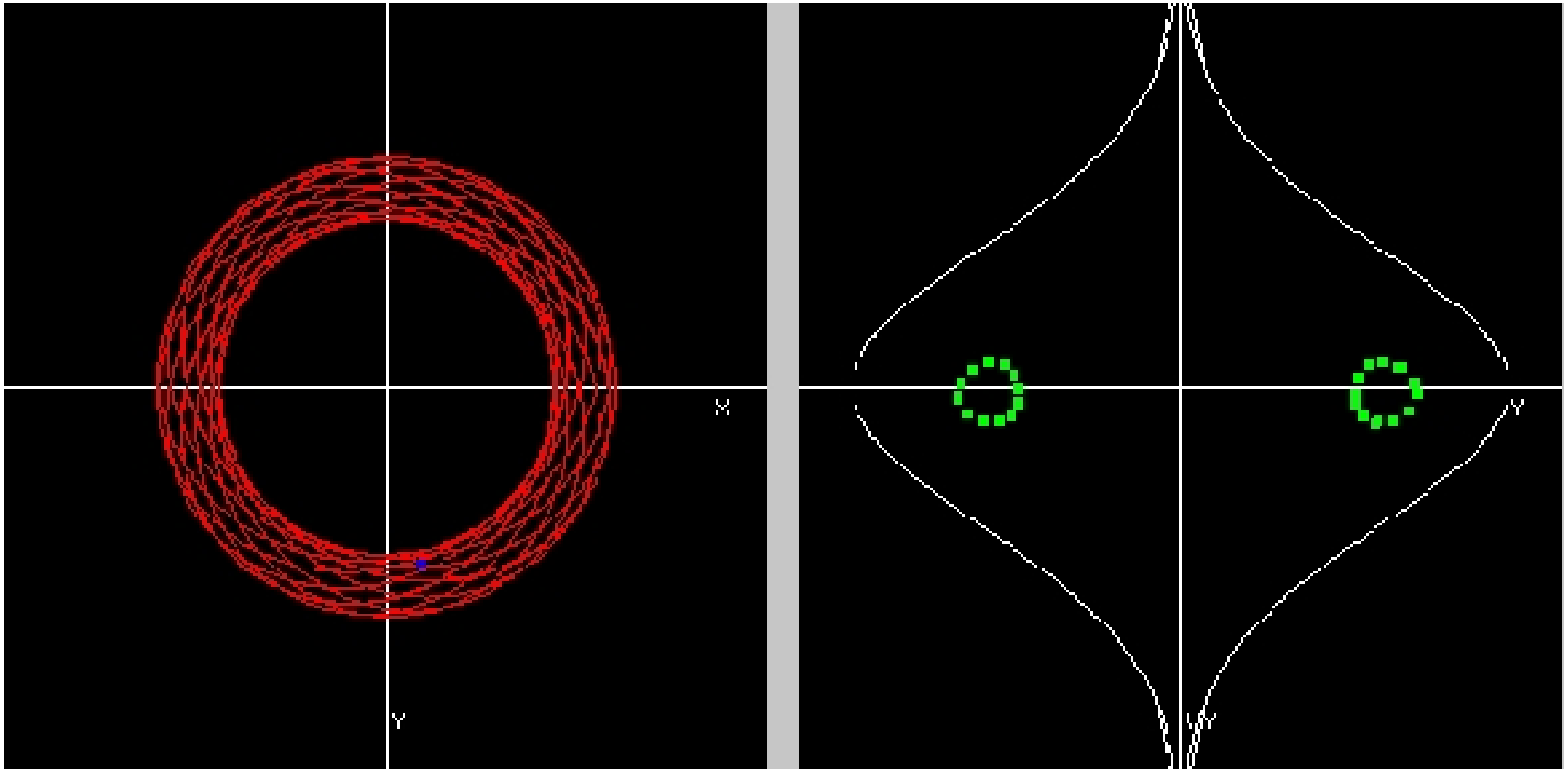
Example from the Binney & Tremaine (see figs. 3-2, 3-3 and 3-4):

$$\Phi = \frac{1}{2} v_0^2 \ln\left(R^2 + \frac{z^2}{q^2}\right) \tag{35}$$

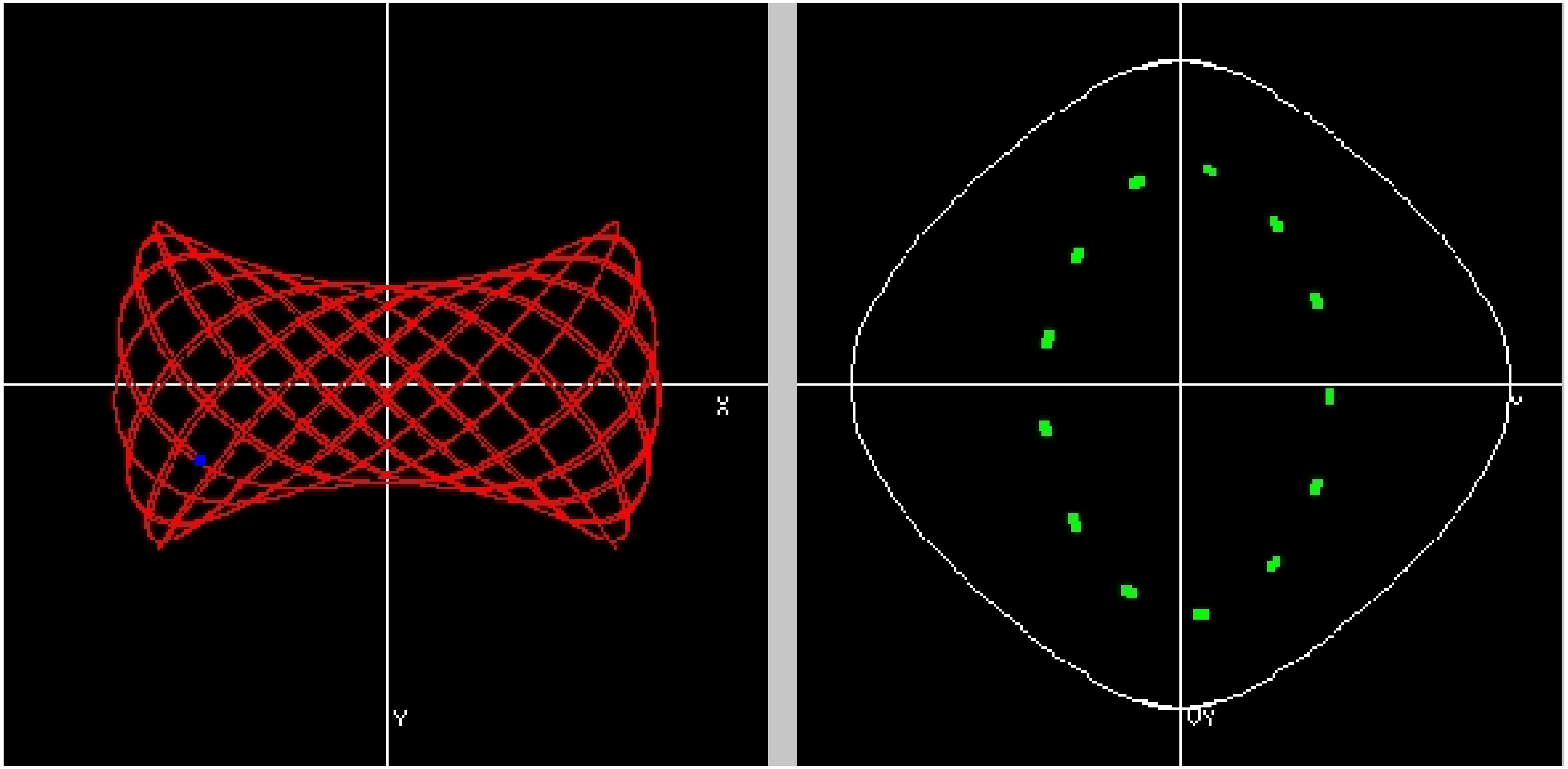


(a)
.2
.1
0
−.1
−.2
Z
0
.1
.2
.3
.4
.5
R
(b)
.1
.05
0
−.05
−.1
Z
0
.1
2
.3
.4
.5
R

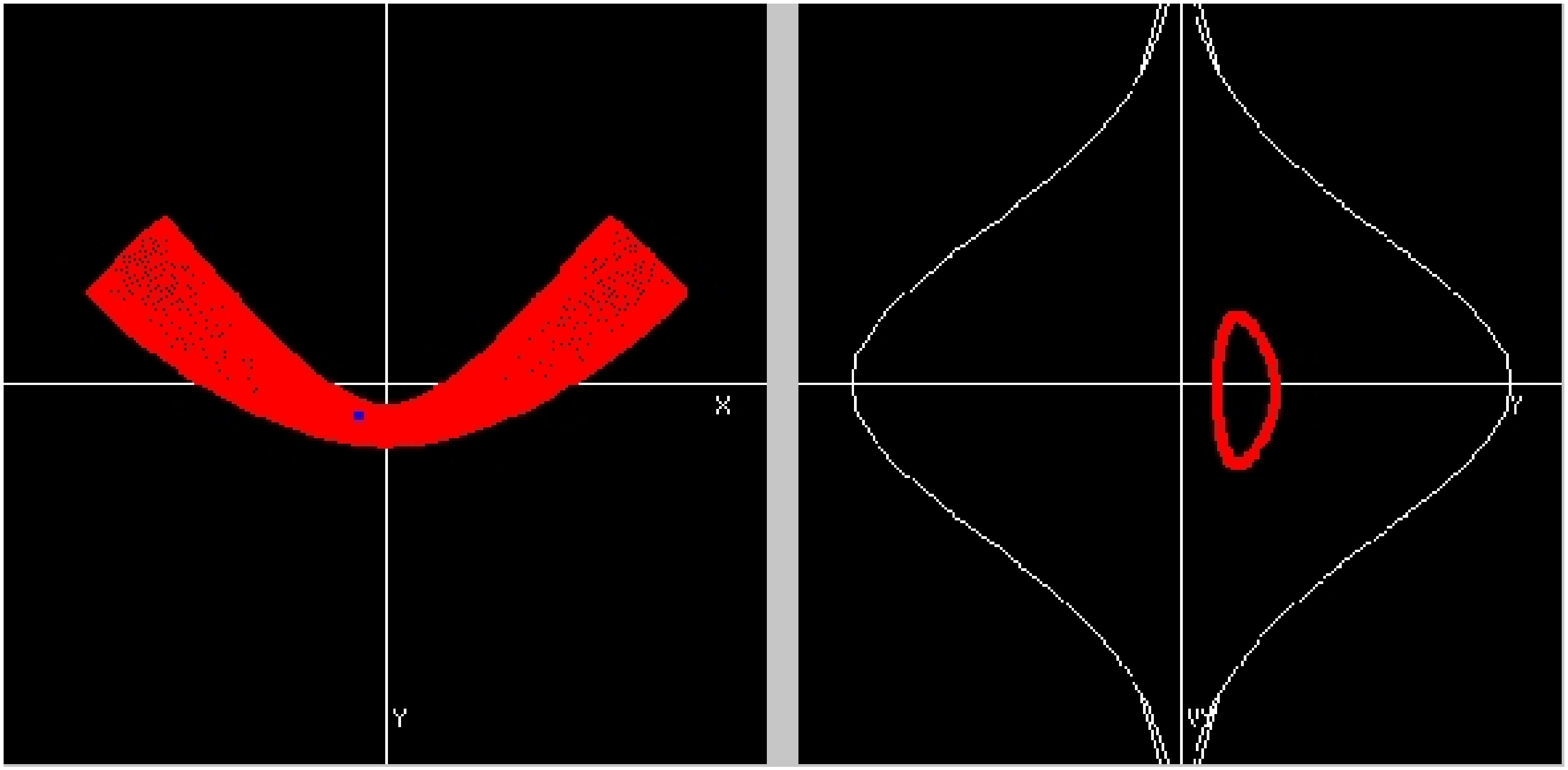
Loop Orbit



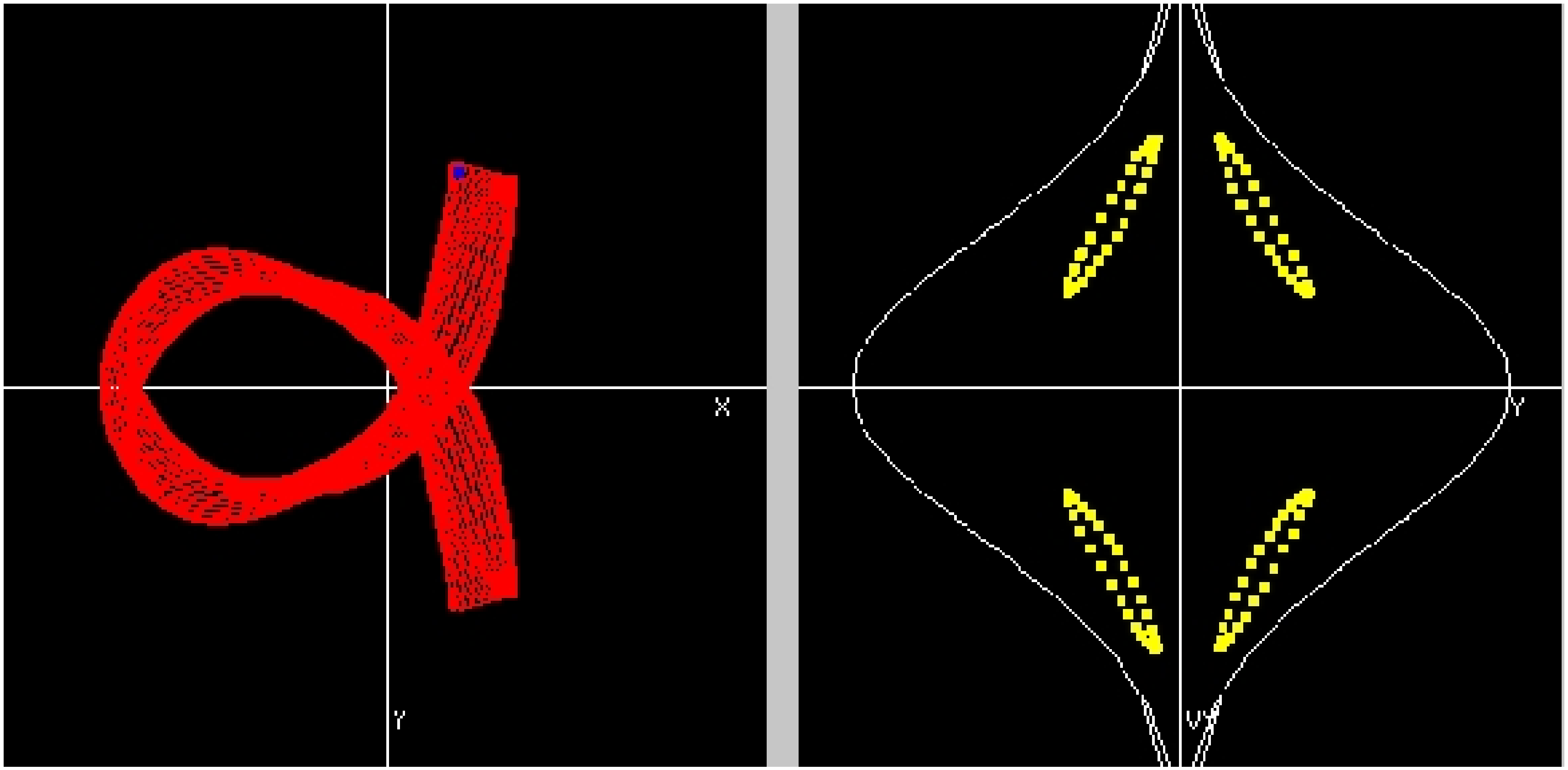
Box Orbit



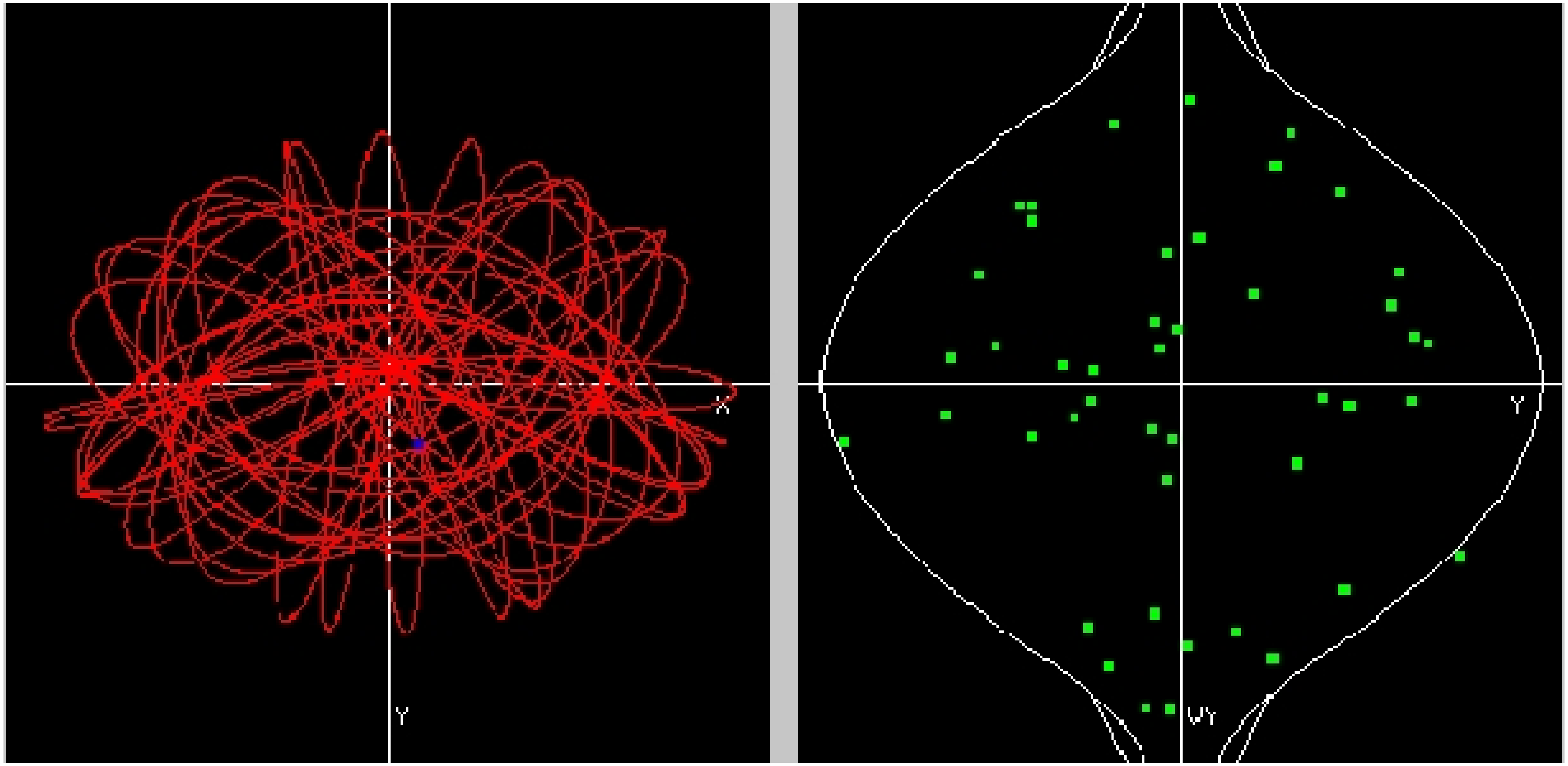
Banana Orbit



Fish Orbit



Box Orbit Scattered by a Point Mass



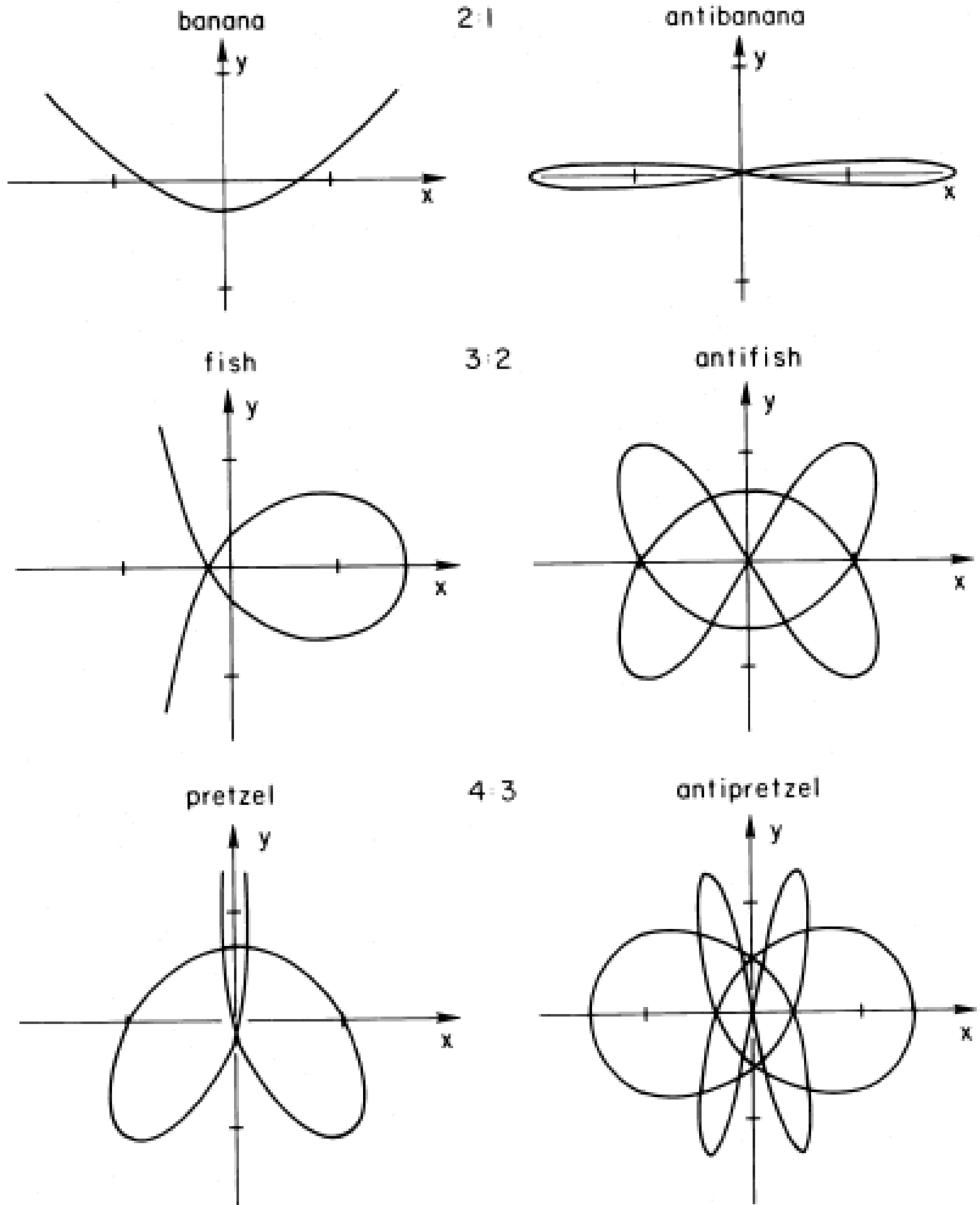
Axisymmetric Potentials

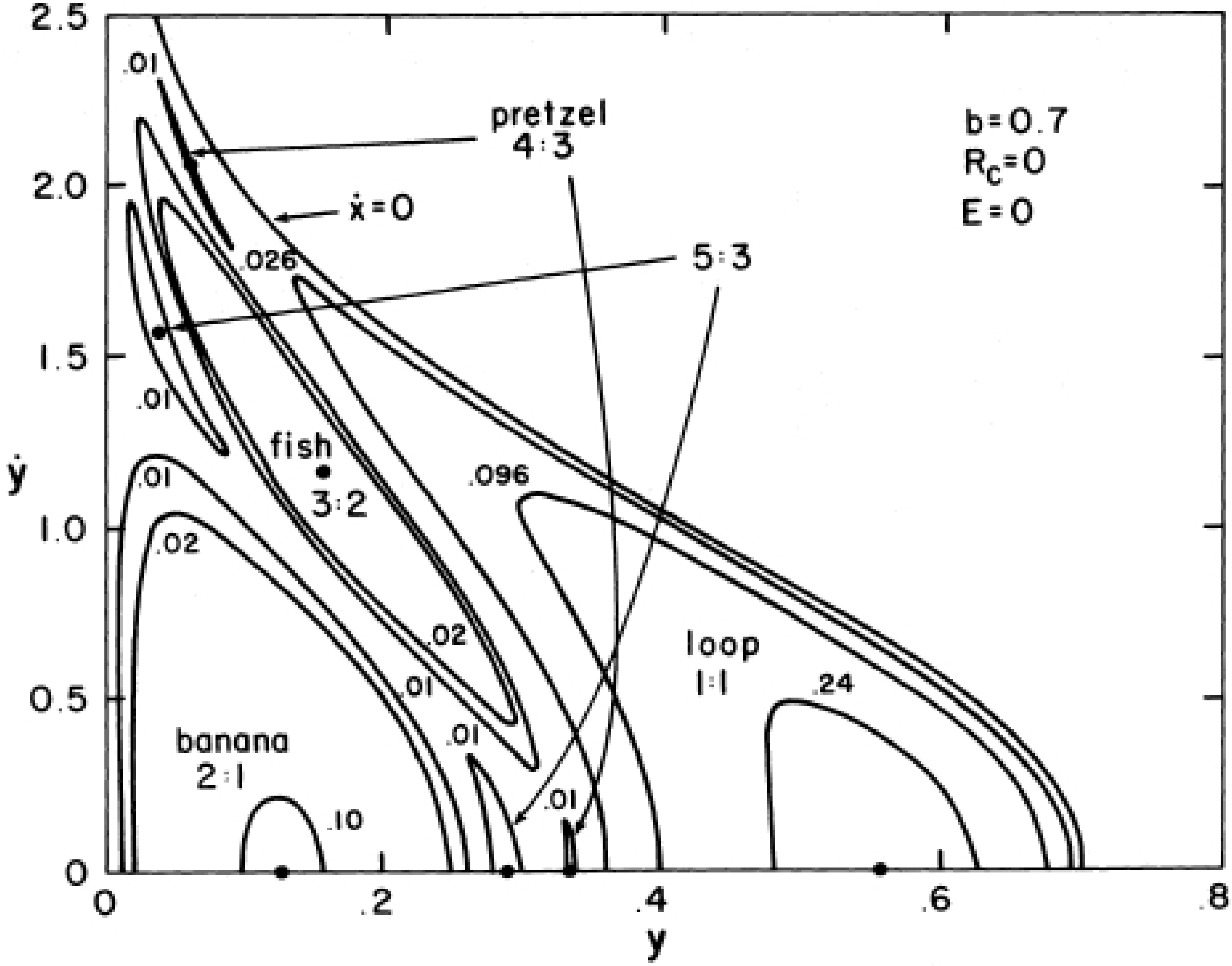
For a modern approach, see Thomas et al. 2004, MNRAS 353, 391: orbit libraries, a Voronoi tessellation of the surface of section, the reconstruction of phase-space distribution function

For a more classic orbital analysis (and if you are interested in finding out what is an "antipretzel"), see Miralda-Escudé & Schwarzschild 1989 (ApJ 339, 752):

Another classic paper is de Zeeuw 1985 (MNRAS 216, 272) (interested in "unstable butterflies"?)

ORBIT STRUCTURE OF LOGARITHMIC POTENTIAL




pretzel
4:3
b=0.7
R_c=0
E=0
$\dot{x}$=0
5:3
.01
.026
.01
fish
3:2
.096
.01
.02
.02
.01
loop
1:1
.24
.01
banana
2:1
.10
.01
$\dot{y}$
y
2.5
2.0
1.5
1.0
0.5
0
0
.2
.4
.6
.8

Epicycle Approximation for Orbits

Assume an axisymmetric potential Φ_{eff} and **nearly circular orbits**; expand Φ_{eff} in a Taylor series about its minimum:

$$\Phi_{\text{eff}} = \text{const} + \frac{1}{2}\kappa^2 x^2 + \frac{1}{2}\nu^2 z^2 + \cdots, \tag{36}$$

where

$$x \equiv R - R_g, \quad \kappa^2 \equiv \left.\frac{\partial^2 \Phi_{\text{eff}}}{\partial R^2}\right|_{(R_g,0)}, \quad \nu^2 \equiv \left.\frac{\partial^2 \Phi_{\text{eff}}}{\partial z^2}\right|_{(R_g,0)}. \tag{37}$$

The equations of motion decouple and we have two integrals:

$$x = X\cos(\kappa t + \phi_0) \qquad z = Z\cos(\nu t + \zeta)$$
$$E_R \equiv \frac{1}{2}[v_R^2 + \kappa^2 (R - R_g)^2] \qquad E_z \equiv \frac{1}{2}[v_z^2 + \nu^2 z^2].$$

Epicycle Approximation

Now compare the **epicycle frequency**, κ, with the angular frequency, Ω.

$$\Omega^2 \equiv \frac{v_c^2}{R^2} = \frac{1}{R}\frac{\partial \Phi}{\partial R} = \frac{1}{R}\frac{\partial \Phi_{\rm eff}}{\partial R} + \frac{L_z^2}{R^4}, \tag{38}$$

$$\kappa^2 = \frac{\partial (R^2\Omega^2)}{\partial R} + \frac{3L_z^2}{R^4} = R\frac{\partial \Omega^2}{\partial R} + 4\Omega^2. \tag{39}$$

Since Ω always decreases, but never faster than Keplerian,

$$\Omega \leq \kappa \leq 2\Omega. \tag{40}$$

These are not the infamous epicycles of Ptolemy's!

Epicycle Approximation

The epicycle approximation also makes a prediction for the ϕ-motion since $L_z = R^2\dot{\phi}$ is conserved. Let

$$y \equiv R_g[\phi - (\phi_0 + \Omega t)] \tag{41}$$

be the displacement in the ϕ direction from the "guiding center". If we expand L_z to first order in displacements from the guiding center, we obtain

$$\phi = \phi_0 + \Omega t - \frac{2\Omega X}{\kappa R_g} \sin(\kappa t + \phi_0). \tag{42}$$

Therefore

$$y = -Y \sin(\kappa t + \phi_0) \quad \text{where} \quad \frac{Y}{X} = \frac{2\Omega}{\kappa} \equiv \gamma \geq 1. \tag{43}$$

$\Rightarrow$ The epicycles are elongated tangentially (for Keplerian motion $\gamma = 2$: **epicycles are not circles** as assumed by Hipparchus and Ptolomey!)

The epicycle frequency (κ) is related to **Oort's constants**:

$$A \equiv \frac{1}{2}\left(\frac{v_c}{R} - \frac{dv_c}{dR}\right)_{R_\odot} = -\frac{1}{2}\left(R\frac{d\Omega}{dR}\right)_{R_\odot} \tag{44}$$

$$B \equiv -\frac{1}{2}\left(\frac{v_c}{R} + \frac{dv_c}{dR}\right)_{R_\odot} = -\left(\frac{1}{2}R\frac{d\Omega}{dR} + \Omega\right)_{R_\odot} = A - \Omega_\odot \tag{45}$$

Then

$$\kappa_\odot^2 = -4B(A-B) = -4B\Omega_\odot \tag{46}$$

In the solar neighborhood,

$$A = 14.5 \pm 1.5 \text{ km/s/kpc}, \quad B = -12 \pm 3 \text{ km/s/kpc}, \tag{47}$$

and so

$$\kappa_\odot = 36 \pm 10 \text{ km/s/kpc}, \tag{48}$$

and

$$\frac{\kappa_\odot}{\Omega_\odot} = 1.3 \pm 0.2 \;\; (> 1 \text{ and } < 2!) \tag{49}$$

For improvements to epicycle approximation see Dehnen 1999 (AJ 118, 1190)

Stellar Dynamics and the Boltzmann Equation

The positions and motions of stars can be described by a phase-space distribution function $f(\mathbf{x}, \mathbf{v}, t)$ (aka the phase-space probability density)

The time evolution of $f(\mathbf{x}, \mathbf{v}, t)$ is described by Newtonian dynamics

Assuming that stars can be neither created nor destroyed, a **continuity equation** can be applied to $f(\mathbf{x}, \mathbf{v}, t)$. In six-dimensional space described by $w_i = (\mathbf{x}, \mathbf{v}) = (x_1, x_2, x_3, v_1, v_2, v_3)$,

$$\frac{\partial f(\mathbf{w}, t)}{\partial t} + \sum_{i=1}^{6} \frac{\partial \left(f(\mathbf{w}, t) \dot{w}_i \right)}{\partial w_i} = 0. \tag{50}$$

The collisonless Boltzmann Equation

$$\frac{\partial(f\dot{w}_i)}{\partial w_i} = \dot{w}_i \frac{\partial f}{\partial w_i} + f \frac{\partial \dot{w}_i}{\partial w_i} \tag{51}$$

Note that the last term is either $(\partial v_i/\partial x_i)$, or $(\partial \dot{v}_i/\partial v_i)$.

This term is always 0: in the first case because v_i and x_i are independent coordinates, and in the second case because $\dot{v}_i = -(\partial \Phi/\partial x_i)$, and Φ does not depend on velocity (because it's gravitational potential). Hence,

$$\frac{\partial f(\mathbf{w},t)}{\partial t} + \sum_{i=1}^{6} \dot{w}_i \frac{\partial f(\mathbf{w},t)}{\partial w_i} = 0. \tag{52}$$

The collisonless Boltzmann Equation (CBE)

$$\frac{\partial f(\mathbf{w},t)}{\partial t} + \sum_{i=1}^{6} \dot{w}_i \frac{\partial f(\mathbf{w},t)}{\partial w_i} = 0. \tag{53}$$

In other forms:

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left[v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} \right] = 0 \tag{54}$$

$$\frac{\partial f}{\partial t} + \mathbf{v} \nabla f = \nabla \Phi \frac{\partial f}{\partial \mathbf{v}} \tag{55}$$

The collisonless Boltzmann Equation (CBE)

The last (vector) notation is the most useful one for expressing the collisonless Boltzmann equation in arbitrary coordinate systems

Very difficult to solve (and hence not terribly useful from that standpoint), but forms the basis for deriving the Jeans equations.

A side note: encounters between stars require another term.

Another side note: the radiative transfer equation is also a special case of the general Boltzmann Equation (in the limit that all particles move at the same speed).

The Moment Equations

Now let us integrate the CBE expressed in form (4) over all velocities:

$$\int \frac{\partial f}{\partial t} \mathrm{d}^3\mathbf{v} + \int v_i \frac{\partial f}{\partial x_i} \mathrm{d}^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} \mathrm{d}^3\mathbf{v} = 0. \tag{56}$$

How do we evaluate these integrals? Two rules:

1. Derivative wrt $\mathbf{x}$, or a function of $\mathbf{x}$, can be taken out

2. Introduce notation

$$\int g(\mathbf{v}) f \mathrm{d}^3\mathbf{v} = < g > \int f \mathrm{d}^3\mathbf{v} \tag{57}$$

   where

$$\nu(\mathbf{x}) = \int f \mathrm{d}^3\mathbf{v} \tag{58}$$

   is the number density as a function of position.

The Moment Equations

Then

$$\int \frac{\partial f}{\partial t} \mathrm{d}^3\mathbf{v} + \int v_i \frac{\partial f}{\partial x_i} \mathrm{d}^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} \mathrm{d}^3\mathbf{v} = 0. \tag{59}$$

with

$$\overline{v}_i \equiv \frac{1}{\nu} \int f v_i \mathrm{d}^3\mathbf{v}, \tag{60}$$

becomes

$$\frac{\partial \nu}{\partial t} + \frac{\partial(\nu \overline{v}_i)}{\partial x_i} = 0. \tag{61}$$

This is just the continuity equation for the stellar number density in real space.

More interesting results are obtained by multiplying the CBE with higher powers of $\mathbf{v}$.

The Moment Equations

E.g. take the first velocity moment of the CBE. Then

$$\int \frac{\partial f}{\partial t} \mathrm{d}^3\mathbf{v} + \int v_i \frac{\partial f}{\partial x_i} \mathrm{d}^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} \mathrm{d}^3\mathbf{v} = 0. \qquad (62)$$

becomes

$$\frac{\partial}{\partial t} \int f v_j \mathrm{d}^3\mathbf{v} + \int v_i v_j \frac{\partial f}{\partial x_i} \mathrm{d}^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i} \int v_j \frac{\partial f}{\partial v_i} \mathrm{d}^3\mathbf{v} = 0. \qquad (63)$$

We can use the divergence theorem to manipulate the last term

$$\int v_j \frac{\partial f}{\partial v_i} \mathrm{d}^3\mathbf{v} = -\int \frac{\partial v_j}{\partial v_i} f \mathrm{d}^3\mathbf{v} = -\int \delta_{ij} f \mathrm{d}^3\mathbf{v} = -\delta_{ij}\nu, \qquad (64)$$

Note that

$$v_j \frac{\partial f}{\partial v_i} = -f \frac{\partial v_j}{\partial v_i} + \frac{\partial (v_j f)}{\partial v_i} \qquad (65)$$

and the last term must be 0 when the integration surface is expendend to infinity (where f must vanish).

The Moment Equations

Eq.(64) can be substituted into (63) giving

$$\frac{\partial(\nu\overline{v_j})}{\partial t} + \frac{\partial(\nu\overline{v_i v_j})}{\partial x_i} + \nu\frac{\partial\Phi}{\partial x_j} = 0, \tag{66}$$

where

$$\overline{v_i v_j} \equiv \frac{1}{\nu}\int v_i v_j f \mathrm{d}^3\mathbf{v}. \tag{67}$$

This is an equation of momentum conservation.

Each velocity can be expressed as a sum of the mean value (aka streaming motion) and the so-called peculiar velocity

$$v_i = \overline{v_i} + w_i \tag{68}$$

where $\overline{w_i} = 0$ by definition.

The Moment Equations

Then

$$\sigma_{ij}^2 \equiv \overline{w_i w_j} = \overline{(v_i - \overline{v}_i)(v_j - \overline{v}_j)} = \overline{v_i v_j} - \overline{v}_i \overline{v}_j. \qquad (69)$$

At each point $\mathbf{x}$ the symmetric tensor σ^2 defines an ellipsoid whose principal axes run parallel to σ^2's eigenvectors and whose semi-axes are proportional to the square roots of σ^2's eigenvalues. This is called the **velocity ellipsoid** at $\mathbf{x}$.

(for an example, see page 27 in Lecture 9)

The Jeans Equations

The continuity equation:

$$\frac{\partial \nu}{\partial t} + \frac{\partial (\nu \bar{v}_i)}{\partial x_i} = 0. \tag{70}$$

and the momentum equation

$$\nu \frac{\partial \overline{v_j}}{\partial t} + \nu \overline{v_i} \frac{\partial \overline{v_j}}{\partial x_i} = -\nu \frac{\partial \Phi}{\partial x_j} - \frac{\partial (\nu \sigma_{ij}^2)}{\partial x_i} \tag{71}$$

The term $-\nu\sigma_{ij}^2$ is a **stress tensor** – it describes an anisotropic pressure.

Note that **the system is not closed**: there is no "equation of state"! The multiplication by higher powers of $\mathbf{v}$ doesn't help – need an *ansatz*. In practice one assumes a particular form for σ_{ij}^2, e.g. for isotropic velocity dispersion $\sigma_{ij}^2 = \sigma^2 \delta_{ij}$

The Jeans Equations

Specialization for an axially symmetric system:

First express the CBE in cylindrical coordinates

$$\frac{\partial f}{\partial t} + \dot{R}\frac{\partial f}{\partial R} + \dot{\phi}\frac{\partial f}{\partial \phi} + \dot{z}\frac{\partial f}{\partial z} + \dot{v}_R\frac{\partial f}{\partial v_R} + \dot{v}_\phi\frac{\partial f}{\partial v_\phi} + \dot{v}_z\frac{\partial f}{\partial v_z} = 0 \qquad (72)$$

With $\dot{R} \equiv v_R$, $\dot{\phi} \equiv v_\phi / R$, and $\dot{z} \equiv v_z$, and

$$\dot{v}_R = -\frac{\partial \Phi}{\partial R} + \frac{v_\phi^2}{R} \qquad (73)$$

$$\dot{v}_\phi = -\frac{1}{R}\frac{\partial \Phi}{\partial \phi} - \frac{v_R v_\phi}{R} \qquad (74)$$

$$\dot{v}_z = -\frac{\partial \Phi}{\partial z} \qquad (75)$$

we get

The Jeans Equations

$$\frac{\partial f}{\partial t} + v_R \frac{\partial f}{\partial R} + v_z \frac{\partial f}{\partial z} + \left[\frac{v_\phi^2}{R} - \frac{\partial \Phi}{\partial R}\right] \frac{\partial f}{\partial v_R} - \frac{v_R v_\phi}{R} \frac{\partial f}{\partial v_\phi} - \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial v_z} = 0 \quad (76)$$

where it was assumed that $\partial/\partial\phi \equiv 0$.

Now we multiply by v_R, v_z and v_ϕ, and integrate over all velocities to get (assuming steady state)

$$\begin{aligned} \frac{\partial(\nu \overline{v_R^2})}{\partial R} + \frac{\partial \nu \overline{v_R v_z}}{\partial z} + \nu \left(\frac{\overline{v_R^2} - \overline{v_\phi^2}}{R} + \frac{\partial \Phi}{\partial R} \right) &= 0, \\ \frac{\partial(\nu \overline{v_R v_\phi}}{\partial R} + \frac{\partial(\nu \overline{v_\phi v_z})}{\partial z} + \frac{2\nu}{R} \overline{v_\phi v_R} &= 0, \qquad (77) \\ \frac{\partial(\nu \overline{v_R v_z})}{\partial R} + \frac{\partial(\nu \overline{v_z^2})}{\partial z} + \frac{\nu \overline{v_R v_z}}{R} + \nu \frac{\partial \Phi}{\partial z} &= 0. \end{aligned}$$

Lovely! And powerful.

Some Applications of the Jeans Equations

- Asymmetric drift
- The local mass density
- The shape of local velocity ellipsoid
- Spheroidal components with isotropic velocity dispersion
- Halo mass density profile

Asymmetric drift

Observations indicate that stars with large $\overline{v_R^2}$ rotate more slowly:

$$\overline{v_\phi} = v_c - \overline{v_R^2}/D \tag{78}$$

with $D \approx 120$ km/s. This can be explained using the v_R Jeans equation.

From the v_R Jeans equation at $z = 0$, with an assumed symmetry around the equatorial plane, $\partial\nu/\partial z = 0$, and definitions $\sigma_\phi^2 = \overline{v_\phi^2} - \overline{v_\phi}^2$ and $v_c^2 = R(\partial\Phi/\partial R)$:

$$\overline{v_\phi} = v_c - \frac{\overline{v_R^2}}{2v_c}\zeta, \tag{79}$$

where

$$\zeta = \frac{\sigma_\phi^2}{\overline{v_R^2}} - 1 - \frac{\partial \ln(\nu\overline{v_R^2})}{\partial \ln R} - \frac{R}{\overline{v_R^2}}\frac{\partial(\overline{v_R v_z})}{\partial z} \tag{80}$$

How large is each of these terms?

Asymmetric drift

$$\zeta = \frac{\sigma_\phi^2}{\overline{v_R^2}} - 1 - \frac{\partial \ln(\nu \overline{v_R^2})}{\partial \ln R} - \frac{R}{\overline{v_R^2}} \frac{\partial(\overline{v_R v_z})}{\partial z} \tag{81}$$

1. We know that locally $\overline{v_z^2}/\overline{v_R^2} \approx \sigma_\phi^2/\overline{v_R^2} \approx 0.45$

2. $R(\partial(\overline{v_R v_z})/\partial z)/\overline{v_R^2}$ is somewhere between 0 and 0.55

3. The largest term is $\partial \ln(\nu \overline{v_R^2})/\partial \ln R \approx 2(\partial \ln \nu/\partial \ln R) \approx R_\odot/R_d \approx 2.4$, where it was assumed that $v_R^2 \propto \nu$ and that $\nu(R) \propto \exp(-R/R_d)$.

Asymmetric drift

Hence,

$$\zeta = 0.45 - 1 - 4.8 - x = -5.35 - x \tag{82}$$

where $0 < x < 0.55$. That is, ζ is uncertain to within only 10%.

These arguments can be inverted, and the measured value of ζ (from asymmetric drift slope) can be used to infer $R_\odot/R_d$ (or, more generally, $\partial \ln \nu/\partial \ln R$).

If there were no density gradient, there would be no asymmetric drift!