Lecture 8: Equilibria of Collisionless Systems. II

The Jeans Equations

The collisonless Boltzmann Equation

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left[v_i\frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i}\right] = 0 \tag{1}$$

or

$$\frac{\partial f}{\partial t} + \mathbf{v}\nabla f = \nabla\Phi\frac{\partial f}{\partial \mathbf{v}} \tag{2}$$

where

$$f \equiv f(\mathbf{x}, \mathbf{v}) \tag{3}$$

Introduce the summing convention, so that

$$\frac{\partial f}{\partial t} + v_i\frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i} = 0 \tag{4}$$

The Moment Equations

Now let us integrate the CBE expressed in form (4) over all velocities:

$$\int \frac{\partial f}{\partial t} \mathrm{d}^3\mathbf{v} + \int v_i \frac{\partial f}{\partial x_i} \mathrm{d}^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} \mathrm{d}^3\mathbf{v} = 0. \tag{5}$$

How do we evaluate these integrals? Two rules:

1. Derivative wrt $\mathbf{x}$, or a function of $\mathbf{x}$, can be taken out

2. Introduce notation

$$\int g(\mathbf{v}) f \mathrm{d}^3\mathbf{v} = < g > \int f \mathrm{d}^3\mathbf{v} \tag{6}$$

where

$$\nu(\mathbf{x}) = \int f \mathrm{d}^3\mathbf{v} \tag{7}$$

is the number density as a function of position.

The Moment Equations

Then

$$\int \frac{\partial f}{\partial t} \mathrm{d}^3\mathbf{v} + \int v_i \frac{\partial f}{\partial x_i} \mathrm{d}^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} \mathrm{d}^3\mathbf{v} = 0. \tag{8}$$

with

$$\overline{v}_i \equiv \frac{1}{\nu} \int f v_i \mathrm{d}^3\mathbf{v}, \tag{9}$$

becomes

$$\frac{\partial \nu}{\partial t} + \frac{\partial (\nu \overline{v}_i)}{\partial x_i} = 0. \tag{10}$$

This is just the continuity equation for the stellar number density in real space.

More interesting results are obtained by multiplying the CBE with higher powers of $\mathbf{v}$.

The Moment Equations

E.g. take the first velocity moment of the CBE. Then

$$\int \frac{\partial f}{\partial t} \mathrm{d}^3\mathbf{v} + \int v_i \frac{\partial f}{\partial x_i} \mathrm{d}^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} \mathrm{d}^3\mathbf{v} = 0. \tag{11}$$

becomes

$$\frac{\partial}{\partial t} \int f v_j \mathrm{d}^3\mathbf{v} + \int v_i v_j \frac{\partial f}{\partial x_i} \mathrm{d}^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i} \int v_j \frac{\partial f}{\partial v_i} \mathrm{d}^3\mathbf{v} = 0. \tag{12}$$

We can use the divergence theorem to manipulate the last term

$$\int v_j \frac{\partial f}{\partial v_i} \mathrm{d}^3\mathbf{v} = -\int \frac{\partial v_j}{\partial v_i} f \mathrm{d}^3\mathbf{v} = -\int \delta_{ij} f \mathrm{d}^3\mathbf{v} = -\delta_{ij}\nu, \tag{13}$$

Note that

$$v_j \frac{\partial f}{\partial v_i} = -f \frac{\partial v_j}{\partial v_i} + \frac{\partial (v_j f)}{\partial v_i} \tag{14}$$

and the last term must be 0 when the integration surface is expendend to infinity (where f must vanish).

The Moment Equations

Eq.(13) can be substituted into (12) giving

$$\frac{\partial(\nu\overline{v_j})}{\partial t} + \frac{\partial(\nu\overline{v_i v_j})}{\partial x_i} + \nu\frac{\partial \Phi}{\partial x_j} = 0, \tag{15}$$

where

$$\overline{v_i v_j} \equiv \frac{1}{\nu}\int v_i v_j f \mathrm{d}^3\mathbf{v}. \tag{16}$$

This is an equation of momentum conservation.

Each velocity can be expressed as a sum of the mean value (aka streaming motion) and the so-called peculiar velocity

$$v_i = \overline{v_i} + w_i \tag{17}$$

where $\overline{w_i} = 0$ by definition.

The Moment Equations

Then

$$\sigma_{ij}^2 \equiv \overline{w_i w_j} = \overline{(v_i - \overline{v}_i)(v_j - \overline{v}_j)} = \overline{v_i v_j} - \overline{v}_i \overline{v}_j. \tag{18}$$

At each point $\mathbf{x}$ the symmetric tensor σ^2 defines an ellipsoid whose principal axes run parallel to σ^2's eigenvectors and whose semi-axes are proportional to the square roots of σ^2's eigenvalues. This is called the **velocity ellipsoid** at $\mathbf{x}$.

The Jeans Equations

The continuity equation:

$$\frac{\partial \nu}{\partial t} + \frac{\partial(\nu \bar{v}_i)}{\partial x_i} = 0. \tag{19}$$

and the momentum equation

$$\nu \frac{\partial \overline{v_j}}{\partial t} + \nu \overline{v_i} \frac{\partial \overline{v_j}}{\partial x_i} = -\nu \frac{\partial \Phi}{\partial x_j} - \frac{\partial(\nu \sigma_{ij}^2)}{\partial x_i} \tag{20}$$

The term $-\nu\sigma_{ij}^2$ is a **stress tensor** – it describes an anisotropic pressure.

Note that the system is not closed: there is no "equation of state"! The multiplication by higher powers of $\mathbf{v}$ doesn't help – need an *ansatz*. In practice one assumes a particular form for σ_{ij}^2, e.g. for isotropic velocity dispersion $\sigma_{ij}^2 = \sigma^2 \delta_{ij}$

The Jeans Equations

Specialization for an axially symmetric system:

First express the CBE in cylindrical coordinates

$$\frac{\partial f}{\partial t} + \dot{R}\frac{\partial f}{\partial R} + \dot{\phi}\frac{\partial f}{\partial \phi} + \dot{z}\frac{\partial f}{\partial z} + \dot{v}_R\frac{\partial f}{\partial v_R} + \dot{v}_\phi\frac{\partial f}{\partial v_\phi} + \dot{v}_z\frac{\partial f}{\partial v_z} = 0 \qquad (21)$$

With $\dot{R} \equiv v_R$, $\dot{\phi} \equiv v_\phi / R$, and $\dot{z} \equiv v_z$, and

$$\dot{v}_R = -\frac{\partial \Phi}{\partial R} + \frac{v_\phi^2}{R} \qquad (22)$$

$$\dot{v}_\phi = -\frac{1}{R}\frac{\partial \Phi}{\partial \phi} - \frac{v_R v_\phi}{R} \qquad (23)$$

$$\dot{v}_z = -\frac{\partial \Phi}{\partial z} \qquad (24)$$

we get

The Jeans Equations

$$\frac{\partial f}{\partial t}+v_R\frac{\partial f}{\partial R}+v_z\frac{\partial f}{\partial z}+\left[\frac{v_\phi^2}{R}-\frac{\partial\Phi}{\partial R}\right]\frac{\partial f}{\partial v_R}-\frac{v_R v_\phi}{R}\frac{\partial f}{\partial v_\phi}-\frac{\partial\Phi}{\partial z}\frac{\partial f}{\partial v_z}=0 \quad (25)$$

where it was assumed that $\partial/\partial\phi \equiv 0$.

Now we multiply by v_R, v_z and v_ϕ, and integrate over all velocities to get (assuming steady state)

$$\begin{aligned}
\frac{\partial(\nu\overline{v_R^2})}{\partial R}+\frac{\partial\nu\overline{v_R v_z}}{\partial z}+\nu\left(\frac{\overline{v_R^2}-\overline{v_\phi^2}}{R}+\frac{\partial\Phi}{\partial R}\right) &= 0,\\
\frac{\partial(\nu\overline{v_R v_\phi}}{\partial R}+\frac{\partial(\nu\overline{v_\phi v_z})}{\partial z}+\frac{2\nu}{R}\overline{v_\phi v_R} &= 0, \qquad (26)\\
\frac{\partial(\nu\overline{v_R v_z})}{\partial R}+\frac{\partial(\nu\overline{v_z^2})}{\partial z}+\frac{\nu\overline{v_R v_z}}{R}+\nu\frac{\partial\Phi}{\partial z} &= 0.
\end{aligned}$$

Lovely! And powerful.

Some Applications of the Jeans Equations

- Asymmetric drift

- The local mass density

- The shape of local velocity ellipsoid

- Spheroidal components with isotropic velocity dispersion

Asymmetric drift

Observations indicate that stars with large $\overline{v_R^2}$ rotate more slowly:

$$\overline{v_\phi} = v_c - \overline{v_R^2}/D \tag{27}$$

with $D \approx 120$ km/s. This can be explained using the v_R Jeans equation.

Limit the consideration to $z = 0$, assume $\partial\nu/\partial z = 0$, define $\sigma_\phi^2 = \overline{v_\phi^2} - \overline{v_\phi}^2$, and note that $v_c^2 = R(\partial\Phi/\partial R)$, to get

$$\overline{v_\phi} = v_c - \frac{\overline{v_R^2}}{2v_c}\zeta, \tag{28}$$

where

$$\zeta = \frac{\sigma_\phi^2}{\overline{v_R^2}} - 1 - \frac{\partial \ln(\nu\overline{v_R^2})}{\partial \ln R} - \frac{R}{\overline{v_R^2}}\frac{\partial(\overline{v_R v_z})}{\partial z} \tag{29}$$

Asymmetric drift

Assuming that $v_c \approx 220$ km/s, we are done if we can convince ourselves that $\zeta \approx 2v_c/D \approx 2.6$. Can we?

1. Locally $\overline{v_z^2}/\overline{v_R^2} \approx \sigma_\phi^2/\overline{v_R^2} \approx 0.45$

2. For $(\partial \ln(\nu\overline{v_R^2})/\partial \ln R)$ assume that $\overline{v_R^2} \propto \nu$ because observations indicate that $\overline{v_z^2} \propto \nu$

3. $R(\partial(\overline{v_R v_z})/\partial z)/\overline{v_R^2}$ is somewhere between 0 and 0.55 – this is significantly smaller than other terms

Hence, ζ does seem to be confined to a very narrow range of plausible values, and thus $v_a \equiv \overline{v_\phi} - v_c \propto \overline{v_R^2}$.