Astr 509: Astrophysics III: Stellar Dynamics

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Lecture 7: Equilibria of Collisionless Systems. I

The Collisionless Boltzmann Equation

A Touch of Fluid Mechanics

The Continuity Equation: consider a fluid described by its density, $\rho(\mathbf{x}, t)$, and assume that no fluid is destroyed, or added to the flow.

Choose an arbitrary closed volume, V, fixed in position and shape, and bounded by a surface S. Then,

The mass of fluid in this volume is

$$M(t) = \int_V \rho(\mathbf{x}, t) d^3\mathbf{x} \tag{1}$$

This mass changes with time at a rate

$$\frac{dM(t)}{dt} = \int_V \frac{\partial \rho(\mathbf{x}, t)}{\partial t} d^3\mathbf{x} \tag{2}$$

The Continuity Equation

The mass of fluid flowing through the surface element per unit time is $\rho(\mathbf{x},t)\mathbf{v}d^2\mathbf{S}$, and thus

$$\int_V \frac{\partial\rho(\mathbf{x},t)}{\partial t} d^3\mathbf{x} + \int_S \rho(\mathbf{x},t)\mathbf{v}d^2\mathbf{S} = 0 \tag{3}$$

With the aid of the divergence theorem

$$\int_V \left[\frac{\partial\rho(\mathbf{x},t)}{\partial t} + \nabla(\rho(\mathbf{x},t)\mathbf{v})\right] d^3\mathbf{x} = 0 \tag{4}$$

Since this result must hold for any volume, we finally get

$$\frac{\partial\rho(\mathbf{x},t)}{\partial t} + \nabla\left(\rho(\mathbf{x},t)\mathbf{v}\right) = 0. \tag{5}$$

Note that $\nabla(\rho\mathbf{v}) = \sum_i(\partial\rho v_i/\partial v_i)$.

Connection to Stellar Dynamics

The positions and motions of stars can be described by a phase-space distribution function $f(\mathbf{x}, \mathbf{v}, t)$ (aka the phase-space probability density)

The time evolution of $f(\mathbf{x}, \mathbf{v}, t)$ is described by Newtonian dynamics

Assuming that stars can be neither created nor destroyed, a **continuity equation** can be applied to $f(\mathbf{x}, \mathbf{v}, t)$. In six-dimensional space described by $w_i = (\mathbf{x}, \mathbf{v}) = (x_1, x_2, x_3, v_1, v_2, v_3)$,

$$\frac{\partial f(\mathbf{w}, t)}{\partial t} + \sum_{i=1}^{6} \frac{\partial \left(f(\mathbf{w}, t)\dot{w}_i\right)}{\partial w_i} = 0. \tag{6}$$

The collisonless Boltzmann Equation

$$\frac{\partial(f\dot{w}_i)}{\partial w_i} = \dot{w}_i\frac{\partial f}{\partial w_i} + f\frac{\partial \dot{w}_i}{\partial w_i} \tag{7}$$

Note that the last term is either $(\partial v_i/\partial x_i)$, or $(\partial \dot{v}_i/\partial v_i)$.

This is always 0: in the first case because v_i and x_i are independent coordinates, and in the second case because $\dot{v}_i = -(\partial\Phi/\partial x_i)$, and Φ does not depend on velocity (because it's gravitational potential). Hence,

$$\frac{\partial f(\mathbf{w},t)}{\partial t} + \sum_{i=1}^{6} \dot{w}_i\frac{\partial f(\mathbf{w},t)}{\partial w_i} = 0. \tag{8}$$

The collisonless Boltzmann Equation

$$\frac{\partial f(\mathbf{w},t)}{\partial t} + \sum_{i=1}^{6} \dot{w}_i \frac{\partial f(\mathbf{w},t)}{\partial w_i} = 0. \tag{9}$$

In other forms:

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left[v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} \right] = 0 \tag{10}$$

$$\frac{\partial f}{\partial t} + \mathbf{v} \nabla f = \nabla \Phi \frac{\partial f}{\partial \mathbf{v}} \tag{11}$$

The collisonless Boltzmann Equation

The last (vector) notation is the most useful one for expressing the collisonless Boltzmann equation in arbitrary coordinate systems

Very difficult to solve (and hence not terribly useful from that standpoint), but forms the basis for deriving the Jeans equations – to be discussed next time.

Encounters between stars require another term – to be discussed later.

A side note: the radiative transfer equation is also a special case of the general Boltzmann Equation (in the limit that all particles move at the same speed).