Astr 509: Astrophysics III: Stellar Dynamics

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Lecture 6: The Orbits of Stars

Axisymmetric Potentials

Axisymmetric Potentials

The problem: given the initial conditions $\mathbf{x}(t_o)$ and $\dot{\mathbf{x}}(t_o)$, and the potential $\Phi(R, z)$, find $\mathbf{x}(t)$.

A better description of real galaxies than spherical potentials, and the orbital structure is much more interesting.

- Poisson's equation for axisymmetric potentials, meridional plane
- Surfaces of Section
- Examples (non-axisymmetric!)
- Epicycle approximation

Axisymmetric Potentials

The equations of motion in an axisymmetric potential (cylindrical coordinates) are

$$\ddot{R} = -\frac{\partial \Phi_{\mathrm{eff}}}{\partial R} \tag{1}$$

and

$$\ddot{z} = -\frac{\partial \Phi_{\mathrm{eff}}}{\partial z} \tag{2}$$

where

$$\Phi_{\mathrm{eff}} \equiv \Phi + \frac{L_z^2}{2R^2} \tag{3}$$

Also

$$\frac{\mathrm{d}}{\mathrm{d}t}(R^2\dot{\phi}) = 0 \quad \Rightarrow \quad R^2\dot{\phi} = L_z. \tag{4}$$

Axisymmetric Potentials

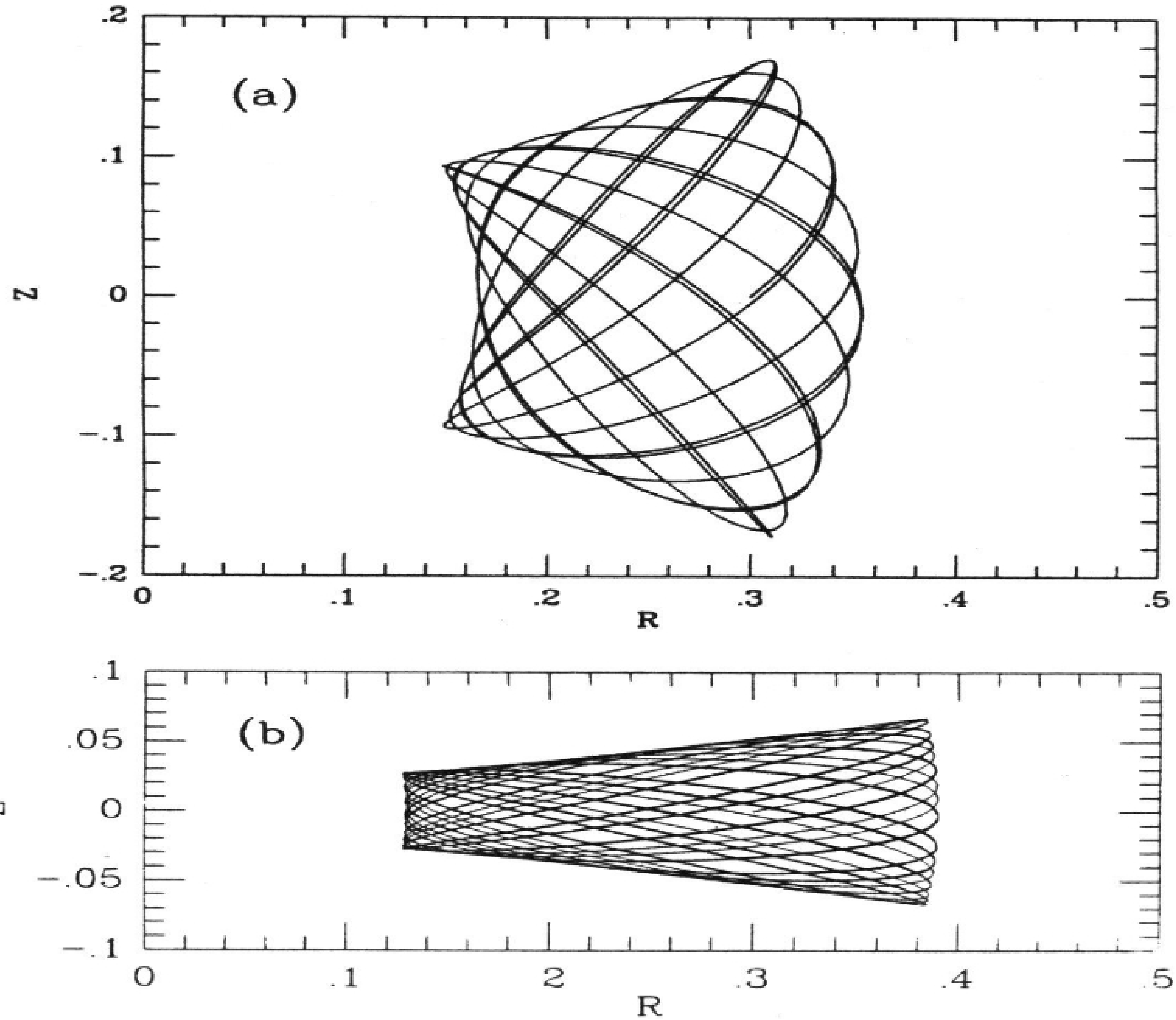
Hence, if solve the first two equations, the solution for ϕ can be obtained from the third equation as

$$\phi(t) = \phi(t_o) + \dot{\phi}(t_o)\, R^2(t_o) \int_{t_o}^{t} dt'/R^2(t') \tag{5}$$

Meridional plane: non-uniformly rotating plane. The three-dimensional motion in the cylindrical (R, z, ϕ) space is reduced to a two-dimensional problem in **Cartesian** coordinates R and z.

Example from the textbook (see figs. 3-2, 3-3 and 3-4).

$$\Phi = \frac{1}{2} v_0^2 \ln \left(R^2 + \frac{z^2}{q^2} \right) \tag{6}$$


(a)
.2
.1
0
−.1
−.2
Z
0
.1
.2
.3
.4
.5
R
(b)
.1
.05
0
−.05
−.1
Z
0
.1
.2
.3
.4
.5
R

Surfaces of Section

In the spherical or nearly spherical case, the **third integral** can be found analytically (in addition to E and L_z)

In the general 2-D case, we can use a graphical device: Poincaré's surface of section.

1. Choose an energy condition

2. Choose a coordinate condition (e.g. $x = 0$ or $z = 0$)

3. Integrate the orbit for given initial conditions and potential

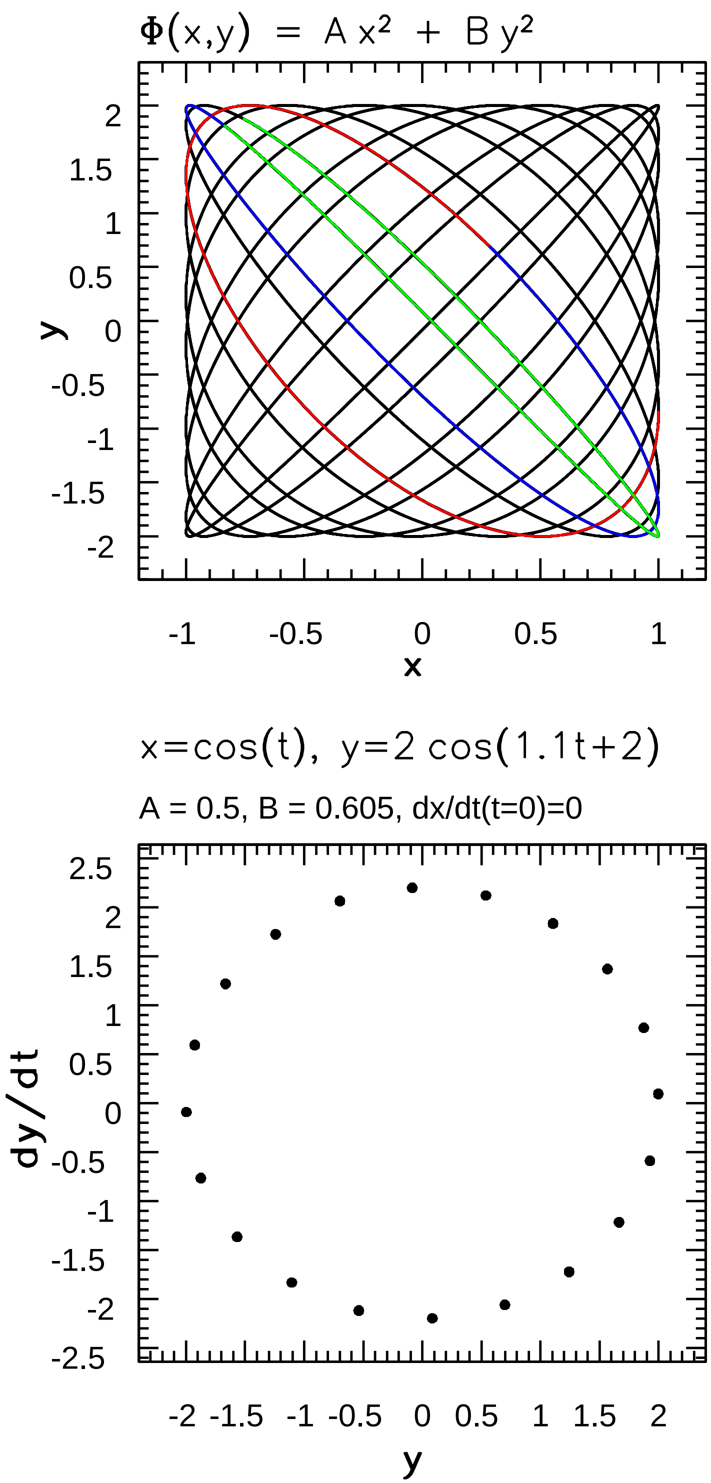
4. Plot the other coordinate vs. its conjugate momentum (the consequent) whenever the coordinate condition is satisfied, e.g. $\dot{y}$ vs. y, or v_R vs. R

Surfaces of Section

- If the orbit is not restricted by another integral, the consequents will fill an area.

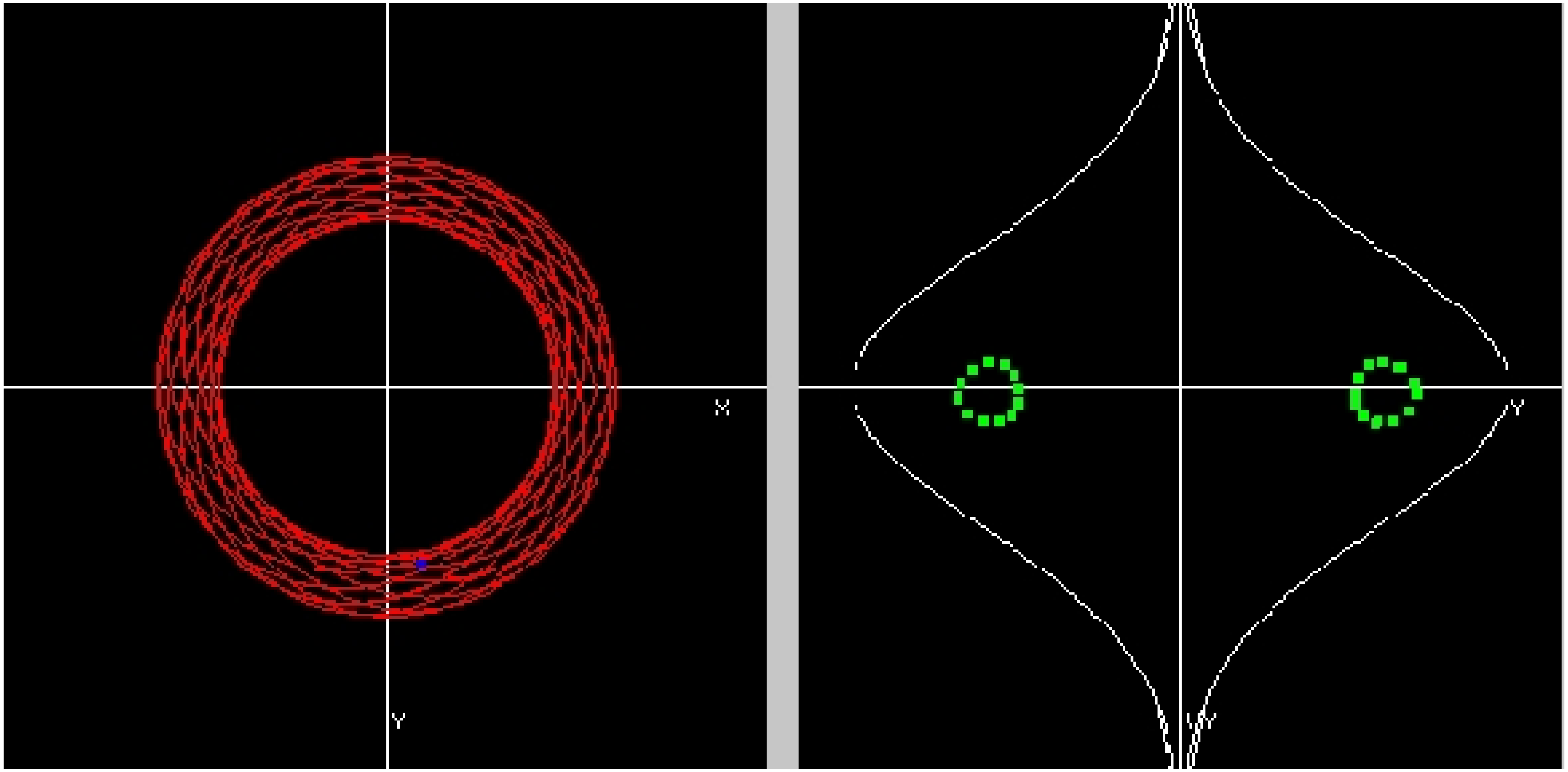
- If the orbit is restricted by another integral, the consequents will lie on a curve.

Example: non-axisymmetric harmonic oscillator

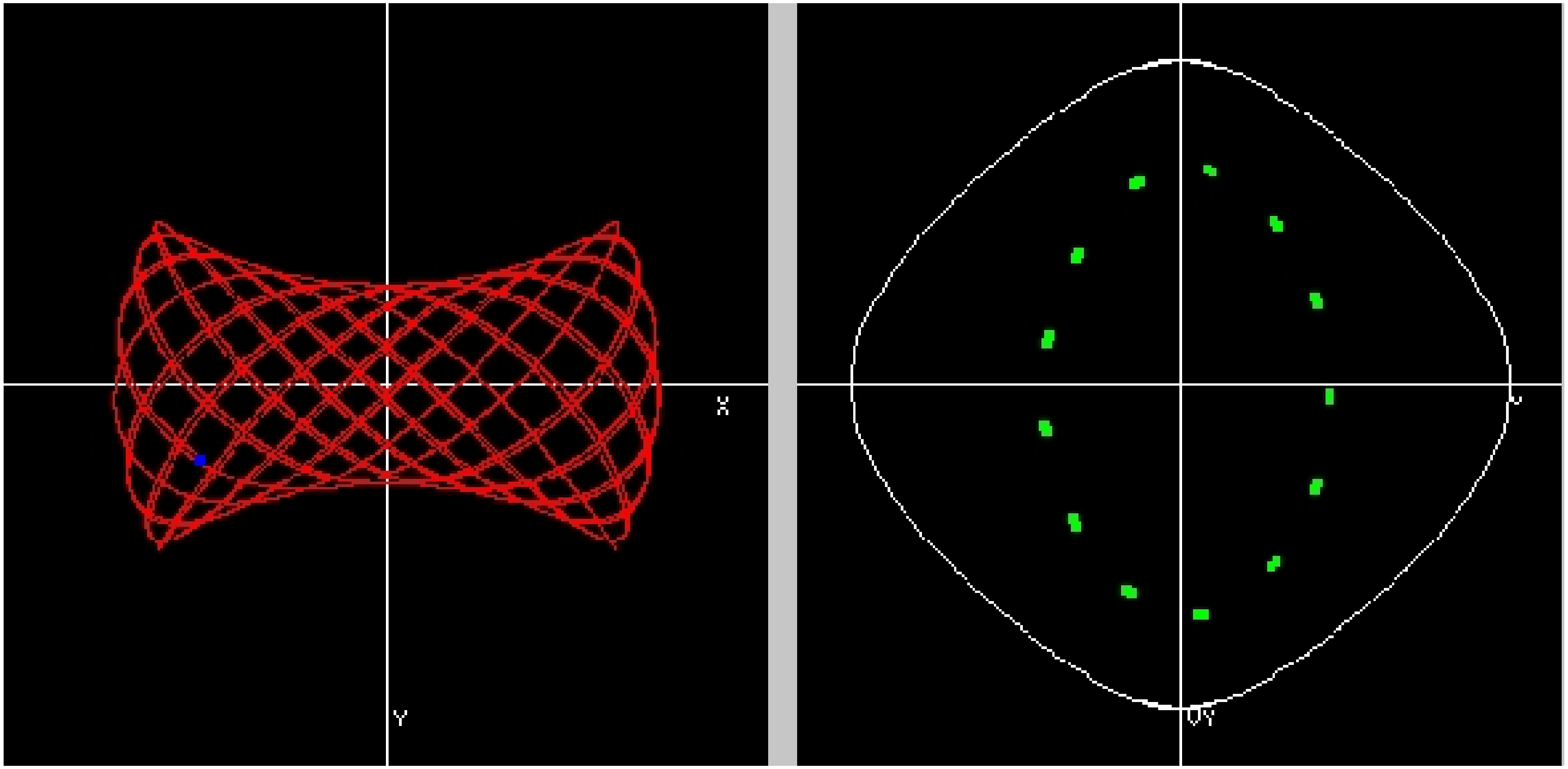


- $\Phi(x,y) = A\,x^2 + B\,y^2$
- For $A = B$ a single ellipse centered on the origin. Here a finite number of orbits because $A/B = n/m = 10/11$. In general, an infinite number of **box** orbits which fill the whole box
- Surface of section: bottom panel, $\dot{y}$ vs y for $x = 0$

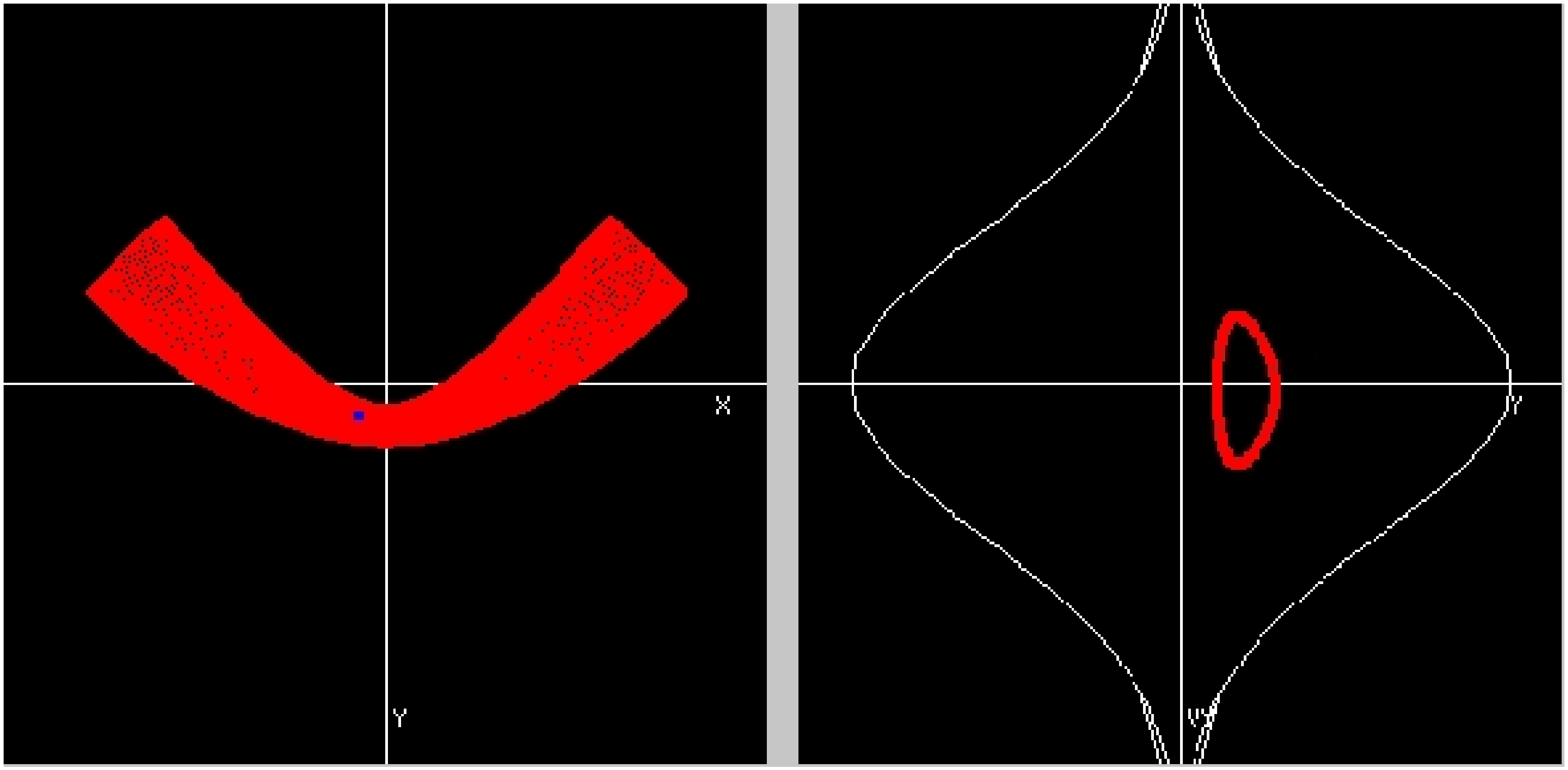
Loop Orbit



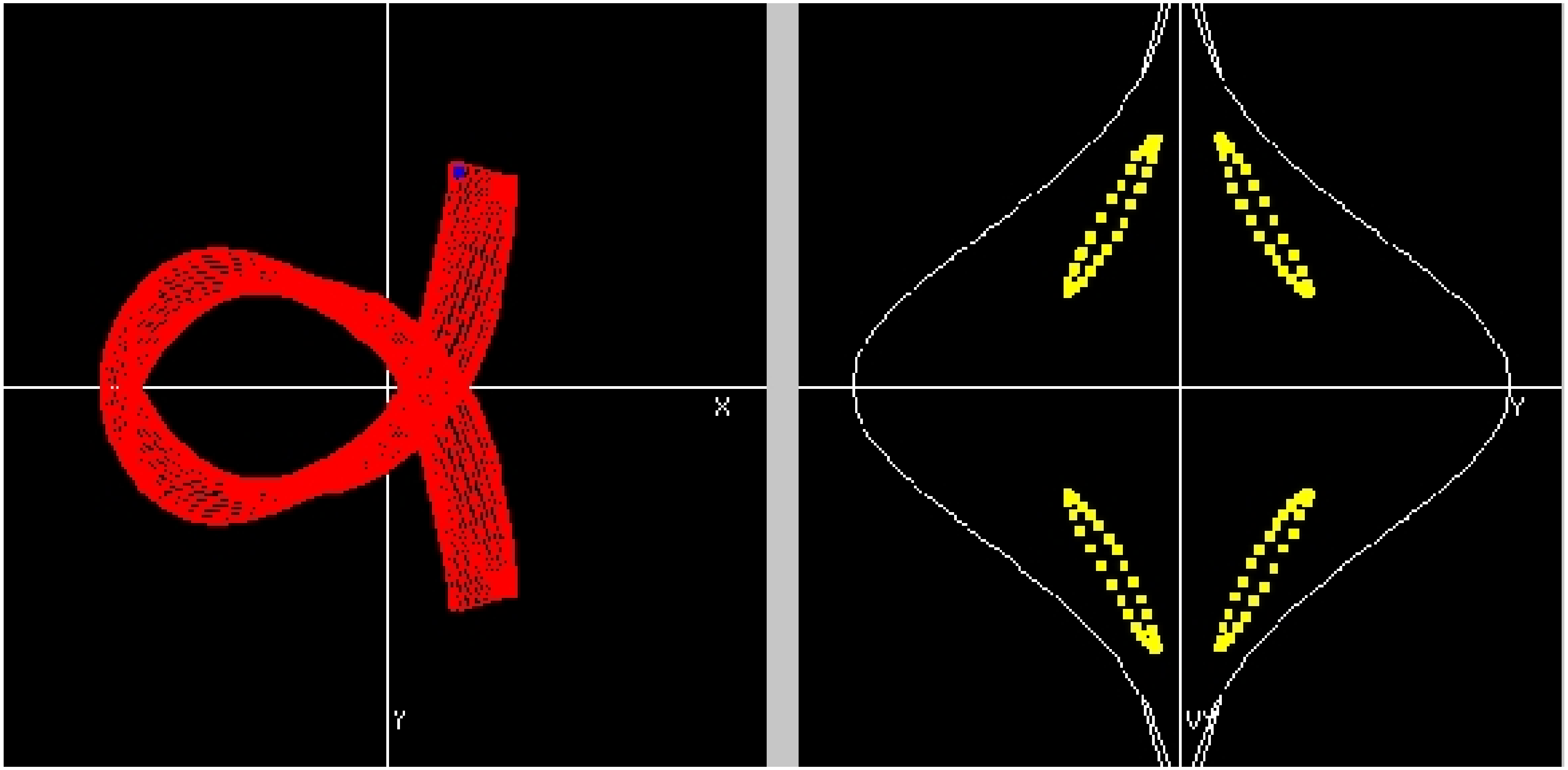
Box Orbit



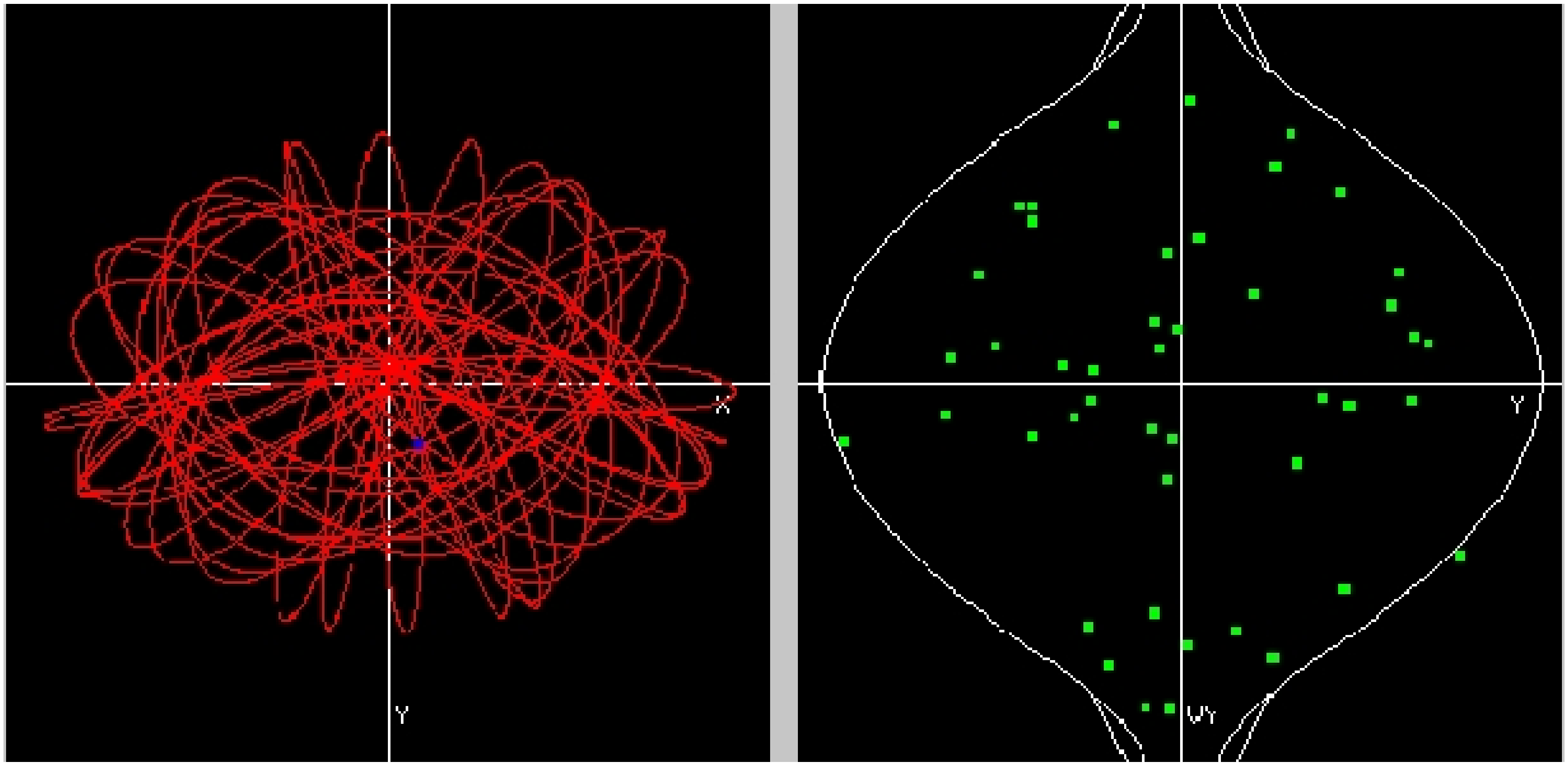
Banana Orbit



Fish Orbit



Box Orbit Scattered by a Point Mass



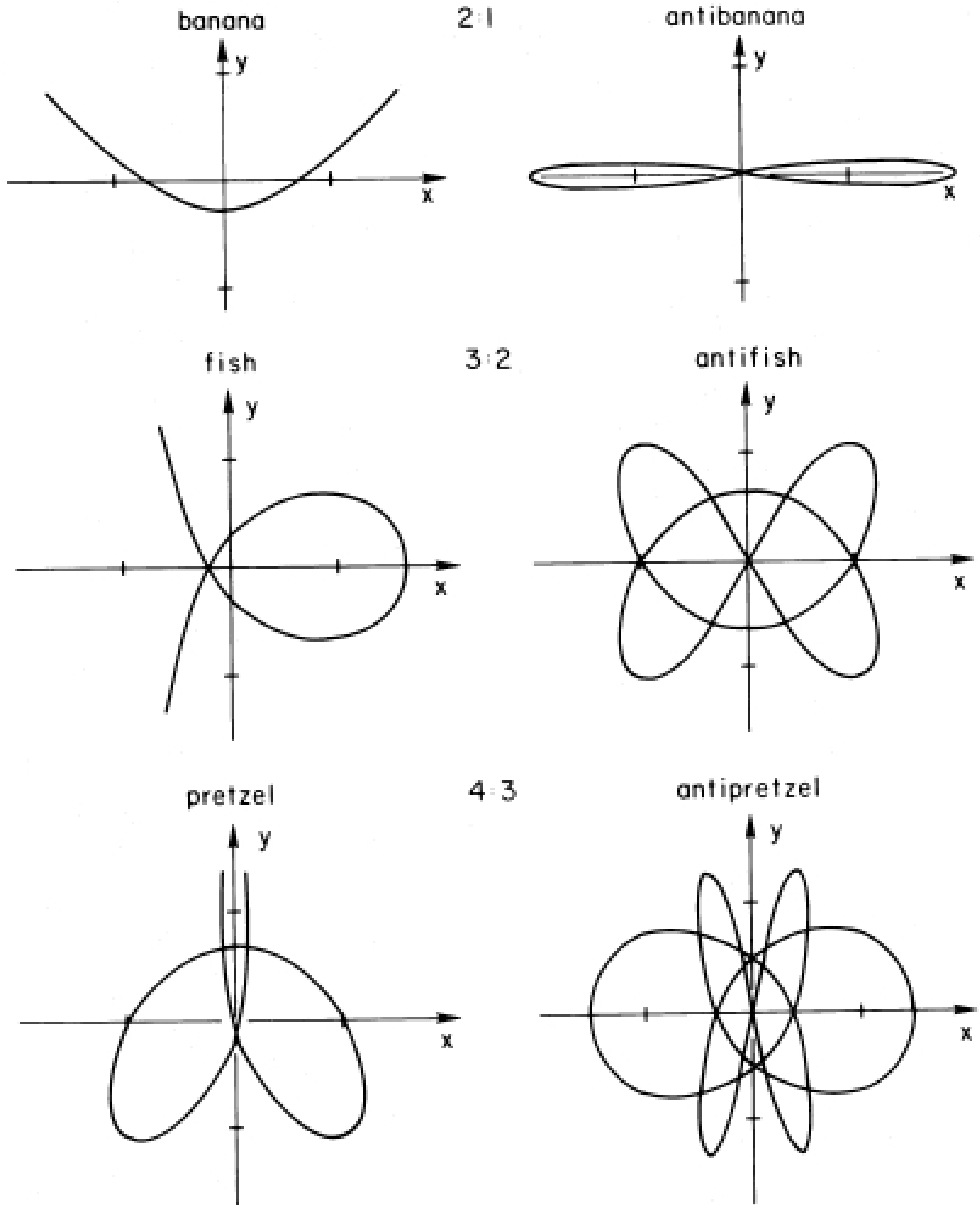
Axisymmetric Potentials

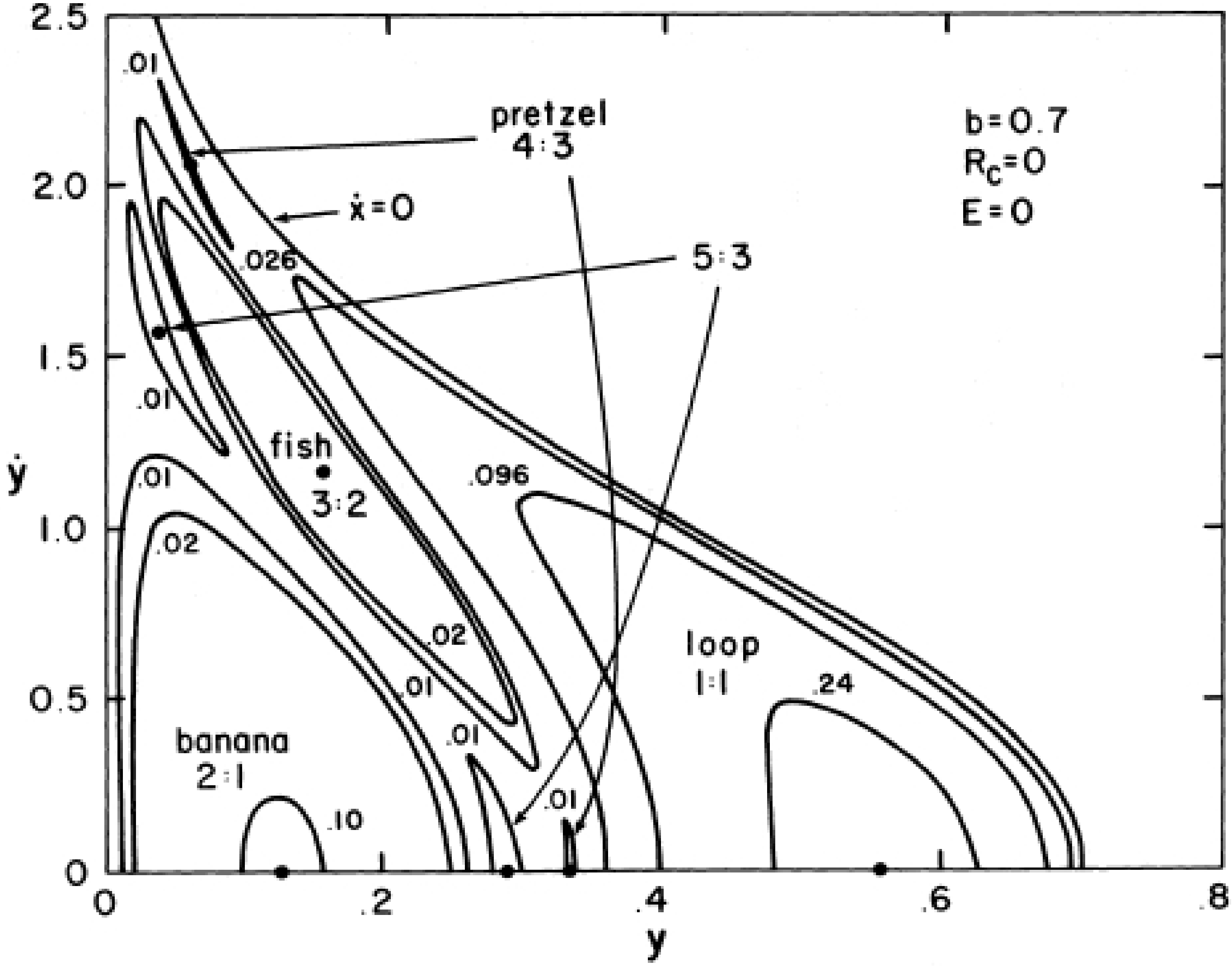
For a modern approach, see Thomas et al. 2004, MNRAS 353, 391: orbit libraries, a Voronoi tessellation of the surface of section, the reconstruction of phase-space distribution function

For a more classic orbital analysis (and if you are interested in finding out what is an “antipretzel”), see Miralda-Escudé & Schwarzschild 1989 (ApJ 339, 752):

Another classic paper is de Zeeuw 1985 (MNRAS 216, 272) (interested in “unstable butterflies”?)

ORBIT STRUCTURE OF LOGARITHMIC POTENTIAL





2.5
2.0
1.5
1.0
0.5
0
ẏ
0
.2
.4
.6
.8
y
.01
pretzel
4:3
b=0.7
R_c=0
E=0
ẋ=0
.026
5:3
.01
fish
3:2
.01
.096
.02
.02
loop
1:1
.01
.24
.01
banana
2:1
.01
.10

Epicycle Approximation

Assume an axisymmetric potential Φ_{eff} and nearly circular orbits; expand Φ_{eff} in a Taylor series about its minimum:

$$\Phi_{\text{eff}} = \text{const} + \frac{1}{2}\kappa^2 x^2 + \frac{1}{2}\nu^2 z^2 + \cdots, \tag{7}$$

where

$$x \equiv R - R_g, \quad \kappa^2 \equiv \left.\frac{\partial^2 \Phi_{\text{eff}}}{\partial R^2}\right|_{(R_g,0)}, \quad \nu^2 \equiv \left.\frac{\partial^2 \Phi_{\text{eff}}}{\partial z^2}\right|_{(R_g,0)}. \tag{8}$$

The equations of motion decouple and we have two integrals:

$$x = X\cos(\kappa t + \phi_0) \qquad z = Z\cos(\nu t + \zeta)$$

$$E_R \equiv \frac{1}{2}[v_R^2 + \kappa^2 (R - R_g)^2] \qquad E_z \equiv \frac{1}{2}[v_z^2 + \nu^2 z^2].$$

Epicycle Approximation

Now compare the **epicycle frequency**, κ, with the angular frequency, Ω.

$$\Omega^2 \equiv \frac{v_c^2}{R^2} = \frac{1}{R}\frac{\partial\Phi}{\partial R} = \frac{1}{R}\frac{\partial\Phi_{\rm eff}}{\partial R} + \frac{L_z^2}{R^4}, \tag{9}$$

$$\kappa^2 = \frac{\partial(R^2\Omega^2)}{\partial R} + \frac{3L_z^2}{R^4} = R\frac{\partial\Omega^2}{\partial R} + 4\Omega^2. \tag{10}$$

Since Ω always decreases, but never faster than Keplerian,

$$\Omega \le \kappa \le 2\Omega. \tag{11}$$

Epicycle Approximation

The epicycle approximation also makes a prediction for the ϕ-motion since $L_z = R^2\dot{\phi}$ is conserved. Let

$$y \equiv R_g[\phi - (\phi_0 + \Omega t)] \tag{12}$$

be the displacement in the ϕ direction from the "guiding center". If we expand L_z to first order in displacements from the guiding center, we obtain

$$\phi = \phi_0 + \Omega t - \frac{2\Omega X}{\kappa R_g}\sin(\kappa t + \phi_0). \tag{13}$$

Therefore

$$y = -Y\sin(\kappa t + \phi_0) \quad \text{where} \quad \frac{Y}{X} = \frac{2\Omega}{\kappa} \equiv \gamma \geq 1. \tag{14}$$

$\Rightarrow$ The epicycles are elongated tangentially (for Keplerian motion $\gamma = 2$ – epicycles are not circles as assumed by Hipparchus and Ptolomey!)

The epicycle frequency (κ) is related to Oort's constants:

$$A \equiv \frac{1}{2}\left(\frac{v_c}{R} - \frac{dv_c}{dR}\right)_{R_\odot} = -\frac{1}{2}\left(R\frac{d\Omega}{dR}\right)_{R_\odot} \tag{15}$$

$$B \equiv -\frac{1}{2}\left(\frac{v_c}{R} + \frac{dv_c}{dR}\right)_{R_\odot} = -\left(\frac{1}{2}R\frac{d\Omega}{dR} + \Omega\right)_{R_\odot} = A - \Omega_\odot \tag{16}$$

Then

$$\kappa_\odot^2 = -4B(A - B) = -4B\Omega_\odot \tag{17}$$

In the solar neighborhood,

$$A = 14.5 \pm 1.5 \text{ km/s/kpc}, \quad B = -12 \pm 3 \text{ km/s/kpc}, \tag{18}$$

and so

$$\kappa_\odot = 36 \pm 10 \text{ km/s/kpc}, \tag{19}$$

and

$$\frac{\kappa_\odot}{\Omega_\odot} = 1.3 \pm 0.2 \ \ (> 1 \text{ and } < 2!) \tag{20}$$

For improvements to epicycle approximation see Dehnen 1999 (AJ 118, 1190)

MISSED from Chapter 3:

- Non-Axysymmetric Potentials (box orbits, loop orbits, etc)

- Rotating Potentials, Lagrange Points

- Bar Potentials, Lindblad Resonances

- Phase-space structure, Stäckel potentials, Delaunay variables

  ...

Integrals

We define an integral to be a function $I(\mathbf{x},\mathbf{v})$ of the phase-space coordinates such that

$$\left.\frac{\mathrm{d}I}{\mathrm{d}t}\right|_{\text{orbit}} = 0.$$

We do not allow I to depend explicitely on time. In the spherical case, L_x, L_y, L_z, and E are integrals, and any function of integrals is also an integral (for example $|\mathbf{L}|^2$). When we start to count integrals, we are actually looking for the largest number of mutually independent integrals.

We expect each independent integral to impose a constraint, $I =$ constant on the phase space coordinates of the orbit. We start out with a 6-D phase space, and each integral will lower the dimensionality of the orbit by one.

For the Kepler problem we know of four integrals, but the Kepler orbit is a 1-D curve. If we look at it in velocity space, it is still a 1-D curve, so there must be a fifth integral. To find the fifth, consider

$$r = \frac{a(1-e^2)}{1 + e\cos(\psi - \psi_0)}$$

where

$$a \equiv \frac{L^2}{GM(1-e^2)} = -\frac{GM}{2E}$$

is the semi-major axis. Note that $a(E)$ and $e(E, L)$ are integrals. Solving for ψ_0, we find that it is also an integral:

$$\psi_0(\mathbf{x}, \mathbf{v}) = \psi - \arccos\left\{\frac{1}{e}\left[\frac{a}{r}(1-e^2) - 1\right]\right\}.$$

So we've found that the number of integrals is (6 $-$ the dimensionality of the orbit). However the number of integrals can exceed this. Consider

$$\Phi = -GM\left(\frac{1}{r} + \frac{r_0}{r^2}\right).$$

An orbit in this potential creates a rosette, which fills a 2-D area of real space, (and also fills a 2-D area in phase space) yet the potential still has a fifth integral, ψ_0. The equation of motion is now

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\psi^2} + \left(1 - \frac{2GMr_0}{L^2}\right) u = \frac{GM}{L^2}.$$

This is, as in the Kepler case, the equation for a harmonic oscillator, but the frequency is no longer 2π. The solution of which is

$$u = \frac{GM}{L^2}\left[K^2 + e\cos\left(\frac{\psi - \psi_0}{K}\right)\right],$$

where

$$K \equiv 1/\sqrt{1 - \frac{2GMr_0}{L^2}}.$$

So ψ_0 is an integral. To see why it is not isolating, look at the solution for ψ:

$$\psi = \psi_0 + K \arccos\left\{\frac{1}{e}\left[\frac{a}{r}(1 - e^2) - K^2\right]\right\}.$$

But we can always add $2m\pi$ to the value of the arccos, adding $2mK\pi$ to ψ. But K will always be irrational so we can approach any value of ψ. Therefore, ψ_0 imposes no useful constraint on the particle's motion.