Astr 509: Astrophysics III: Stellar Dynamics

Winter Quarter 2005, University of Washington, Željko Ivezić

Lecture 5: The Orbits of Stars

Static Spherical Potentials

Orbits in Static Spherical Potentials

The problem: given the initial conditions $\mathbf{x}(t_o)$ and $\dot{\mathbf{x}}(t_o)$, and the potential $\Phi(r)$, find $\mathbf{x}(t)$.

Orbits in spherical potentials are easy to consider and lead to some important concepts.

- Some general considerations
- Example 1: Spherical harmonic oscillator: $\Phi(r) = A + B\, r^2$
- Example 2: Point mass potential: $\Phi(r) = \frac{-GM}{r}$
- Example 3: Isochrone potential: $\Phi(r) = \frac{-GM}{b+\sqrt{b^2+r^2}}$

General considerations

The initial conditions are 6-dimensional and thus a general solution includes six orbital parameters. (aka *constants of motion*)

The equation of motion in a spherical potential is:

$$\ddot{\mathbf{r}} = F(r)\hat{\mathrm{e}}_r, \tag{1}$$

i.e. the force is always radial!

Crossing through by $\mathbf{r}$, we show that the angular momentum vector, $\mathbf{L} \equiv \mathbf{r} \times \dot{\mathbf{r}}$ is conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times \dot{\mathbf{r}}) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} + \mathbf{r} \times \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = F(r) \times \hat{\mathrm{e}}_r = 0 \tag{2}$$

Therefore, the motion is constrained to the plane perpendicular to $\mathbf{L}$, and can be fully described in cylindrical coordinate system, r and ψ ($\mathbf{v} = \dot{r}\,\hat{\mathrm{e}}_r + r\dot{\psi}\,\hat{\mathrm{e}}_\psi$)

General considerations

The equations of motion in the plane are

$$\begin{aligned}\ddot{r} - r\dot{\psi}^2 &= F(r)\\ 2\dot{r}\dot{\psi} + r\ddot{\psi} &= 0.\end{aligned}$$

The second equation comes from $r^2\dot{\psi} = L =$ const. (note that this is the second Kepler's law!)

$\dot{\psi}$ can be eliminated using $\dot{\psi} = L/r^2$, leading to a one-dimensional equation of motion:

$$\ddot{r} - L^2/r^3 = F(r). \tag{3}$$

This equation motivates a definition of an effective potential

$$-\nabla\Phi_{\rm eff} \equiv F(r) + L^2/r^3, \tag{4}$$

and thus

$$\Phi_{\rm eff}(r) \equiv \Phi(r) + \frac{L^2}{2r^2}. \tag{5}$$

General considerations

The energy per unit mass is

$$E = \frac{1}{2}v^2 + \Phi(r) = \frac{1}{2}(\dot{r}^2 + r^2\dot{\psi}^2) + \Phi(r) = \frac{1}{2}\left(\dot{r}^2 + \Phi_{\text{eff}}(r)\right). \quad (6)$$

For bound orbits r oscillates between an inner radius, or pericenter (r_{min}), and an outer radius, or apocenter (r_{max}). The radial period is

$$T_r = 2\int_{r_{\text{min}}}^{r_{\text{max}}} \left(\sqrt{2[E - \Phi_{\text{eff}}(r)]}\right)^{-1} dr \quad (7)$$

The pericenter and apocenter are the solutions of $\Phi_{\text{eff}}(r) = E$.

General considerations

The azimuthal period is

$$T_\psi = \frac{2\pi}{\Delta\psi} T_r \tag{8}$$

where

$$\Delta\psi = 2L \int_{r_{\text{min}}}^{r_{\text{max}}} \left(r^2 \sqrt{2[E - \Phi_{\text{eff}}(r)]}\right)^{-1} dr \tag{9}$$

The orbit is closed only for $\Delta\psi = k(2\pi)$ – in general case, the orbit forms a rosette.

The orbital precession rate:

$$\Omega_p = \frac{\Delta\psi - 2\pi}{T_r} \tag{10}$$

General considerations

If we eliminate t rather than ψ, then we have an equation for the orbit's shape. In terms of the variable $u \equiv 1/r$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\psi^2} + u = -\frac{F(u)}{L^2 u^2} \quad \Rightarrow \quad \frac{\mathrm{d}^2 u}{\mathrm{d}\psi^2} = \zeta(u). \tag{11}$$

This is a second order differential equation for $u(\psi)$, where $\zeta(u)$ and the initial conditions are presumably specified.

Let's now look at specific examples.

The harmonic potential

$$\Phi = \Phi_0 + \frac{1}{2}\Omega^2 r^2. \tag{12}$$

Generated by homogeneous density distribution.

The motion decouples in cartesian co-ordinates to $\ddot{x} = -\Omega^2 x$ and $\ddot{y} = -\Omega y$, and the solution is:

$$x = X\cos(\Omega t + \phi_x), \quad y = Y\sin(\Omega t + \phi_y), \tag{13}$$

where X, Y, ϕ_x and ϕ_y are arbitrary constants (determined from initial conditions).

This is the equation for an ellipse **centered** on the origin.

Orbits are closed since the periods for x and y oscillations are identical.

Point mass (Keplerian) potential

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\psi^2} + u = \frac{GM}{L^2} \quad \Rightarrow \quad u = \frac{GM}{L^2}[1 + e\cos(\psi - \psi_0)]. \tag{14}$$

This is the equation for an ellipse with one **focus at the origin** and eccentricity e (the first Kepler's law). The semi-major axis is $a = L^2/GM(1 - e^2)$.

The motion is periodic in ψ with period 2π. This gives a closed orbit with

$$T_r = T_\psi = 2\pi\sqrt{\frac{a^3}{GM}} = 2\pi GM(2|E|)^{-3/2} \tag{15}$$

Note that $T^2 \propto a^3$ – the third Kepler's law!

Isochrone Potential

$$\Phi(r) = \frac{-GM}{b + \sqrt{b^2 + r^2}} \tag{16}$$

More extended than point mass, less extended than harmonic potential.

T_r same as for the Keplerian case ($T_r = 2\pi GM(2|E|)^{-3/2}$).

However,

$$\Delta\psi = \pi \left[1 + \frac{L}{\sqrt{L^2 + 4GMb}} \right] \tag{17}$$

i.e. $\pi < \Delta\psi < 2\pi$, and hence the orbits are not closed.