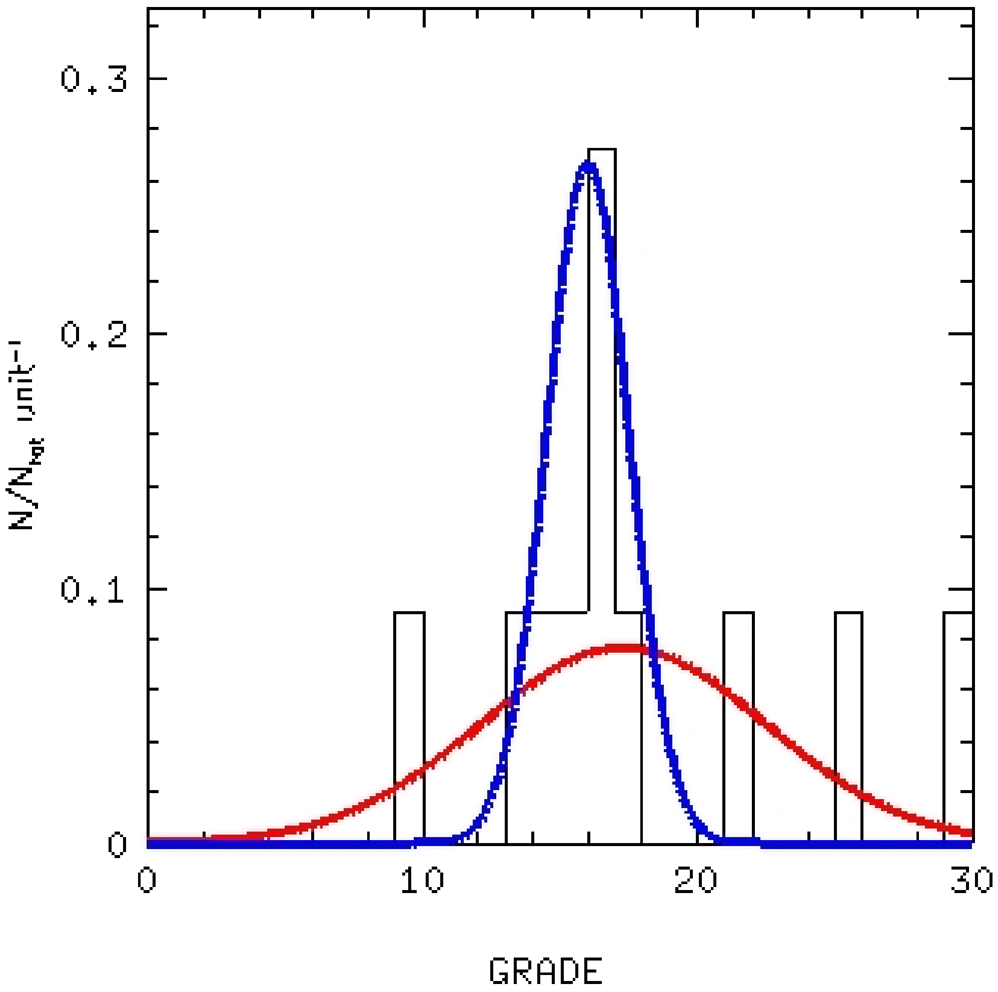
Astr 509: Astrophysics III: Stellar Dynamics

Winter Quarter 2005, University of Washington, Željko Ivezić

Lecture 10: Equilibria of Collisionless Systems

More about the Virial Equations. . .

The Midterm Exam Stats



The Virial Theorem

III. THE VIRIAL THEOREM APPLIED TO CLUSTERS OF NEBULAE

If the total masses of clusters of nebulae were known, the average masses of cluster nebulae could immediately be determined from counts of nebulae in these clusters, provided internebular material is of the same density inside and outside of clusters.

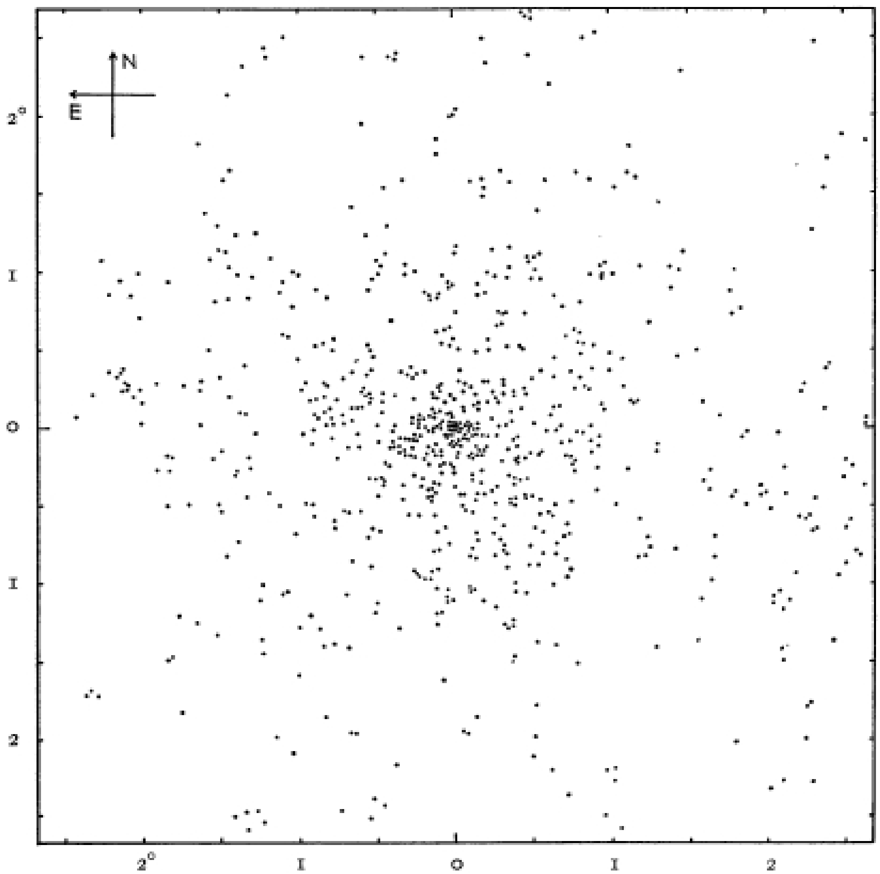
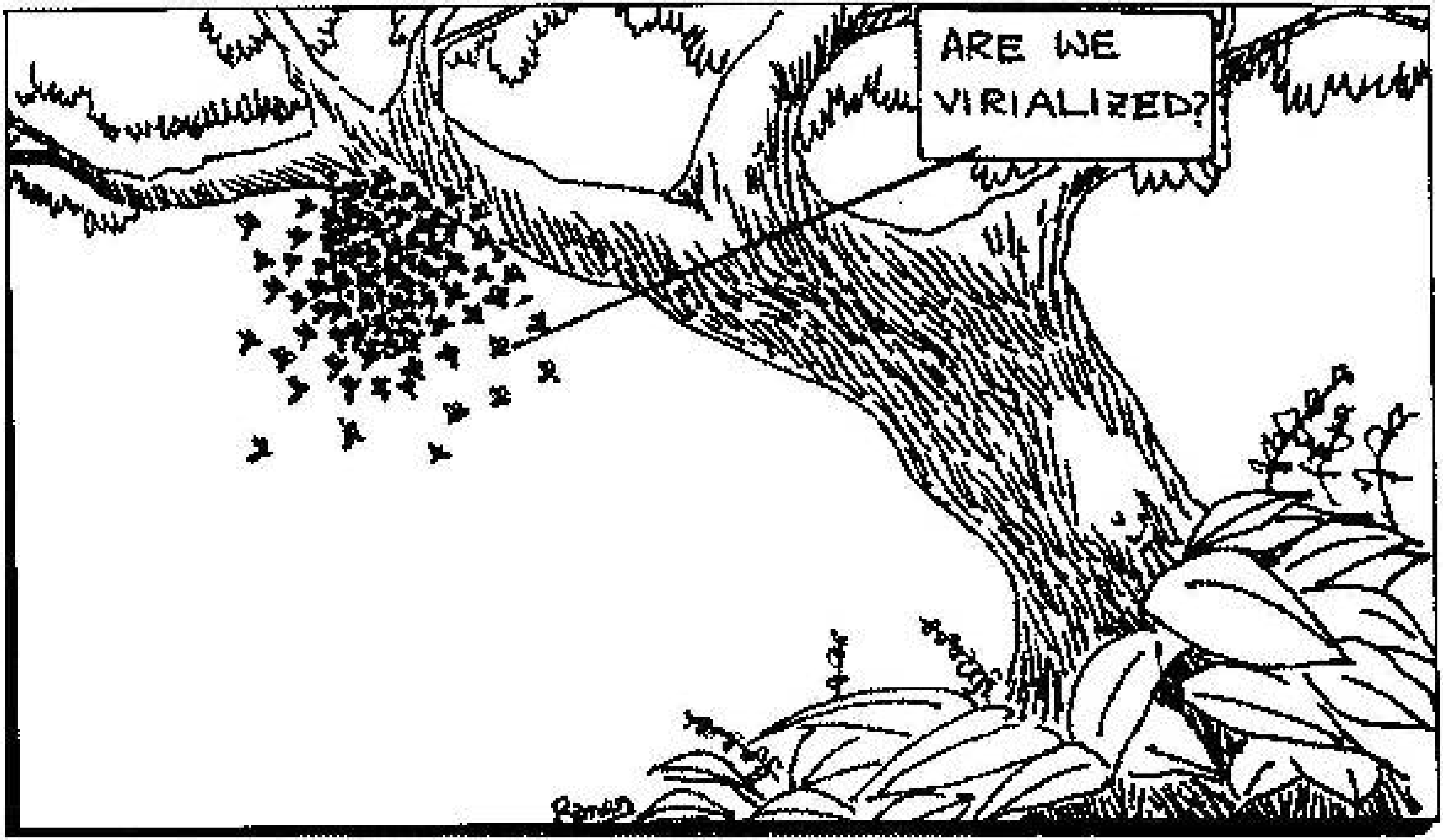


FIG. 3.—The Coma cluster of nebulae

As a first approximation, it is probably legitimate to assume that clusters of nebulae such as the Coma cluster (see Fig. 3) are mechanically stationary systems. With this assumption, the virial theorem of classical mechanics gives the total mass of a cluster in terms of the average square of the velocities of the individual nebulae which constitute this cluster.[5] But even if we drop the assumption that clus-

- In a system of N particles, gravitational forces tend to pull the system together and the stellar velocities tend to make it fly apart. It is possible to relate kinetic and potential energy of a system through the change of its moment of intertia
- In a steady-state system, these tendencies are balanced, which is expressed quantitatively through the the Virial Theorem.
- A system that is not in balance will tend to move towards its virialized state. (see fig. 4.19 in the textbook)



Cartoon courtesy of and ©1999 by B. Nath.

The Virial Theorem(s)

- The Scalar Virial Theorem tells us that the *average* kinetic and potential energy must be in balance.

- The tensor Virial Theorem tells us that the kinetic and potential energy must be in balance in each separate direction.

- The scalar virial theorem is useful for estimating global *average* properties, such as total mass, escape velocity and relaxation time, while the tensor virial theorem is useful for relating shapes of systems to their kinematics, e.g. the flatness of elliptical galaxies to their rotational speed (for a wide range of applications, see Chandrasekhar (1987, Ellipsodial Figures of Equilibrium, New York: Dover).

- Remember that *the virial theorem is a good intuitive tool but one that can be dangerous to put to quantitative use* (King, unpublished).

The Virial Theorem

Zwicki's derivation: (Ap. J. 1937, 86, 217)

$$m_i \frac{d^2 \vec{x}_i}{dt^2} = \vec{F}_i \tag{1}$$

where $\vec{F}_i$ is the total forces on galaxy i.

Scalar multiplication with $\vec{x}_i$ gives:

$$\frac{1}{2} \frac{d^2}{dt^2} (m_i x_i^2) = \vec{x}_i \cdot \vec{F}_i + m_i \left(\frac{d\vec{x}_i}{dt} \right)^2 \tag{2}$$

(summing over all system particles is implied). The term on the left side represents the change of the momentum of inertia, the second term on the right side is related to kinetic energy, and the first term on the right side is called *virial*.

The Virial Theorem

It can be shown (the so-called Euler theorem from classical mechanics) that for $\Phi \propto 1/r$

$$\sum \vec{x}_i \cdot \vec{F}_i = \sum \vec{x}_i \cdot \nabla\Phi = -\Phi \qquad (3)$$

That is, the virial is related to potential energy of the system (true for any *homogeneous* function of the order k such that $\Phi(\lambda x) = \lambda^k \Phi(x)$ – the virial is equal to $k\Phi$).

In a steady state,

$$\frac{1}{2}\frac{d^2}{dt^2}(m_i x_i^2) = 0, \qquad (4)$$

and, for a self-gravitating system in steady state

$$2K + \Phi = 0 \qquad (5)$$

where $K = M < v^2 > /2$ is the kinetic energy. Thus,

$$E = K + \Phi = -K = \frac{1}{2}\Phi \qquad (6)$$

The Scalar Virial Theorem: Applications

- If a system collapses from infinity, half of the potential energy will end up in kinetic energy, and the other half will be disposed of! From the measurement of the circular velocity and the mass of Milky Way (which constrain the kinetic energy), we conclude that during their formation, galaxies radiate away about 3×10^{-7} of their rest-mass energy.

- For a virialized spherical system, $M = 2R\sigma^2/G$. We can estimate total mass from the size and velocity dispersion. E.g. for a cluster with σ=12 km/s, and R=3 pc, we get $M = 2 \times 10^5$ $M_\odot$ (note that $G = 233$ in these units)

The Tensor Virial Theorem

Just as we took velocity moments of the collisionless Boltzmann equation (CBE) to obtain the Jeans equations, we can now take spatial moments of the CBE. If we multiply the CBE by x_k and integrate over space we obtain

$$\int x_k \frac{\partial(\nu \overline{v}_j)}{\mathrm{d}t} \mathrm{d}^3\mathbf{x} = -\int x_k \frac{\partial(\nu \overline{v_i v_j})}{\partial x_i} \mathrm{d}^3\mathbf{x} - \int \nu x_k \frac{\partial \Phi}{\partial x_j} \mathrm{d}^3\mathbf{x}. \qquad (7)$$

The second term on the right hand side can be identified with the **Chandrasekhar potential energy tensor, W**. The first term on the right hand side can be rewritten using the divergence theorem:

$$\int x_k \frac{\partial(\nu \overline{v_i v_j})}{\partial x_i} \mathrm{d}^3\mathbf{x} = -\int \delta_{ki} \nu \overline{v_i v_j} \mathrm{d}^3\mathbf{x} = -2K_{kj}, \qquad (8)$$

where we have defined the **kinetic energy tensor K** by

$$K_{jk} \equiv \frac{1}{2} \int \nu \overline{v_i v_j} \mathrm{d}^3\mathbf{x}. \qquad (9)$$

As with the velocity moments we can split up $\mathbf{K}$ into ordered and random parts:

$$K_{jk} = T_{jk} + \frac{1}{2}\Pi_{jk}, \tag{10}$$

where

$$T_{jk} \equiv \frac{1}{2}\int \nu \overline{v}_j \overline{v}_k \mathrm{d}^3\mathbf{x} \quad ; \quad \Pi_{jk} \equiv \int \nu \sigma_{jk} \mathrm{d}^3\mathbf{x}. \tag{11}$$

If we now symmetrize equation (7) about the indices k and j, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \nu(x_k\overline{v}_j + x_j\overline{v}_k)\mathrm{d}^3\mathbf{x} = 2T_{jk} + \Pi_{jk} + W_{jk}, \tag{12}$$

where we have used the symmetry of $\mathbf{T}$, $\mathbf{\Pi}$ and $\mathbf{W}$ in their indices.

If we now introduce the **moment of inertia tensor** $\mathbf{I}$

$$I_{jk} \equiv \int \nu x_j x_k \mathrm{d}^3\mathbf{x}. \tag{13}$$

The derivative of this with respect to time is

$$\frac{\mathrm{d}I_{jk}}{\mathrm{d}t} = \int \frac{\partial \nu}{\partial t} x_j x_k \mathrm{d}^3\mathbf{x}. \tag{14}$$

We can use continuum equation to change the right side of the equation to

$$-\int \frac{\partial(\nu\bar{v}_i)}{\partial x_i} x_j x_k \mathrm{d}^3\mathbf{x} = \int \nu\bar{v}_i (x_k\delta_{ji} + x_j\delta_{ki}) \mathrm{d}^3\mathbf{x}, \qquad (15)$$

where the second term is obtained using the divergence theorem.

This is now recognizable in the left hand side of (12), so we now have the **tensor virial theorem**

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} I_{jk} = 2T_{jk} + \Pi_{jk} + W_{jk}. \qquad (16)$$

Note that we can get the scalar virial theorem by taking trace of the tensor virial theorem.

The Tensor Virial Theorem: Applications

Example: relating the shapes of elliptical galaxies to their rotational velocity.

From Section 4.3(b)

$$\frac{v_0^2}{\sigma_0^2} = 2(1-\delta)\frac{W_{xx}}{W_{zz}} - 2 \approx \frac{\epsilon}{1-\epsilon} \tag{17}$$

where ϵ is the galaxy's ellipticity, v_0 is the mass-weighted mean rotation speed, σ_0 is the mass-weighted mean random speed along the light of sight, and δ measures the anisotropy of velocity dispersion tensor.

The bottom line is that the **measured** v_0 are much smaller than the values implied by the **measured** ϵ and the above equation – therefore, elliptical galaxies are NOT flattened by rotation!