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Fractional Brownian Motion and Fractional Gaussian Noise

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Abstract. Fractional Brownian motion is one of most cogent mathematical models for strongly correlated stochastic processes with self-similarity. In this article, we give a pedagogic introduction to this theory and investigate some of the statistical, geometric, and fractal properties of fractional Brownian motion and fractional Gaussian random fields. The connection between fractional Brownian motion and the renormalization group in statistical physics is emphasized.

1 Introduction

Random walk and Brownian motion are ubiquitous mathematical models for physical and biological processes [40,5]. The mathematical theory of Brownian motion developed in the early part of last century by Einstein, Kramers, Chandrasekhar, Uhlenbeck, and others has provided physicists and biologists with a powerful tool for analyzing a wide range of natural phenomena. The universal applicability of the model depends on the fact that the systems and processes we have studied in the past often have a large number of uncorrelated or weakly correlated components. However, complex natural structures and processes usually have long-range, strong spatial and temporal correlations. This gives the motivation for studying stochastic processes and random fields with long-range correlation.

The theory of fractional Brownian motion (fBm) [4,20] is a mathematical generalization of the classical theory of random walk and Brownian motion. The term "fractional" is related to fractional integration and differentiation [18]. In contrast to the classical Brownian motion which has independent increments, the fBm has a long-range, strong spatial and temporal correlation as its defining property. Although there is much scholarly work on the rigorous mathematics and statistics of fBm, relatively little has been developed on the applications of fBm as a mechanistic model for physical and biological problems.

The main objective of this article is to present an applied theory for fBm and its related problems. In particular, we introduce fBm according to an elegant approach given in [16,38] which emphasizes its connection with the theory of statistical physics [9,19]. We also present some results on the geometric shape and fractal dimension of fBm.

The concept of scaling invariance and the concept of renormalizing transformation [22,17] are two important ideas in modern statistical physics. They were developed into the theory of renormalization group by K.G. Wilson who was awarded the Nobel Prize in physics in 1982. The idea of scaling gives rise to the notion of critical exponent, or *fractal dimension*, which has become a fertile ground for interdisciplinary sciences [2,6].

2 Self-similarity, Fractional Gaussian Noise, and Fractional Brownian Motion

Fractional Gaussian noise (fGn) and fBm were origianlly introduced by Mandelbrot and van Ness [24] for modeling stochastic fractal processes [23]. The definition for fGn and fBm given by Gallavotti, Jona-Lasinio and independently by Sinai [16,38], however, is more general and insightful from the standpoint of statistical physics [22,17]. In the most general form, an fGn is a random field with E-dimensional random vectors defined in a d-dimensional space. It corresponds to the spin dimension E and space dimension d in the theory of critical phenomena in statistical physics.

It has been shown that the fGn is invariant under a semigroup (Kadanoff block) transformation with critical exponent H, known as the Hurst coefficient in the fields of engineering and stochastic fractal processes [24]. Furthermore, it was shown that other stationary processes approach asymptotically to fGn under the Kadanoff renormalization transformation with respective critical exponents. Hence, the space of all stationary processes is divided under the renormalization group transformation, with fGn as fixed points.

2.1 Self-similarity and fractional Gaussian Noise

A self-similar [38] *fractional Gaussian noise* (fGn) is a series of identical Gaussian random variables $X_1, X_2, X_3, \ldots$, which has the following property [24]

$$\mathcal{A}_N[X] \triangleq \frac{X_1 + X_2 + X_3 + ... + X_N}{N^H} \sim X \tag{1}$$

where '$\sim$' denotes equality in the sense of probability distribution. An alternative but equivalent definition can be given in terms of relative dispersion [1]: $RD_N = N^{H-1} RD_1$ where

$$RD_N = \frac{\sqrt{Var[X_1 + X_2 + ... + X_N]}}{NE[X_1]}.$$

$E[\cdot]$ and $Var[\cdot]$ denote expectation value and variance of a random variable.

It is straightforward to show that when $H = \frac{1}{2}$, the X's are necessarily independent. In general, $0 < H < 1$ and the X's are correlated.

2.2 fBm, fGn and their correlation functions

A *fractional Brownian motion* then is defined as the partial sum of the fGn:

$$B_k^H = X_1 + X_2 + X_3 + ... + X_k.$$

Hence, we have:

$$E[(B_h^H - B_k^H)^2] = E[(h-k)^{2H} X_1^2] = (h-k)^{2H}\sigma^2 \tag{2}$$

where we have assumed, without loss generality, $E[X_1] = 0$ and denoted $\sigma^2 = Var[X_1]$. We can also expand the lhs of Eq. 2 and obtain:

$$E[(B_h^H)^2] - 2E[B_h^H B_k^H] + E[(B_k^H)^2] = h^{2H}\sigma^2 - 2E[B_h^H B_k^H] + k^{2H}\sigma^2 = (h-k)^{2H}\sigma^2.$$

Thus we have the correlation for fBm:

$$E[B_h^H B_k^H] = \frac{\sigma^2}{2}[h^{2H} - (h-k)^{2H} + k^{2H}]. \tag{3}$$

One can also determine the autocorrelation for the fGn:

$$\begin{aligned} E[X_n X_m] &= E[(B_n^H - B_{n-1}^H)(B_m^H - B_{m-1}^H)] \\ &= E[B_n^H B_m^H + B_{n-1}^H B_{m-1}^H - B_{n-1}^H B_m^H - B_n^H B_{m-1}^H] \\ &= \frac{\sigma^2}{2}[(n-m-1)^{2H} - 2(n-m)^{2H} + (n-m+1)^{2H}]. \end{aligned}$$

Using the stationarity of X we have

$$E[X_0 X_k] = \frac{\sigma^2}{2}[(k-1)^{2H} - 2k^{2H} + (k+1)^{2H}]. \tag{4}$$

This result also can be directly verified using the self-similarity [38]:

$$\frac{1}{N^{2H}} E\left[\left(\sum_{\ell=0}^{N-1} X_\ell\right)^2\right] = E[X_0^2] = \sigma^2.$$

One can furthermore verify that:

$$\frac{1}{N^{2H}} E\left[\sum_{s=kN}^{(k+1)N-1} X_s \sum_{t=0}^{N-1} X_t\right] = E[X_0 X_k].$$

That is for any integer $N \geq 1$, autocorrelation function $\rho(k) = E[X_0 X_k]$ uniquely satisfies:

$$\rho(k) = N^{-2H} \sum_{\ell=-(N-1)}^{N-1} (N - |\ell|)\rho(Nk + \ell). \tag{5}$$

For a Gaussian stochastic process, all the joint distributions for B^H's are multivariate Gaussian distributions. For any multivariate Gaussian random variables ξ_1, ξ_2, ..., ξ_N with $E[\xi_k] = 0$ $(k = 1, 2, ..., N)$, the moments with order higher than 2 can all be expressed in terms of the second-order correlation known as the Isserlis theorem or the Wick's theorem in statistical physics [33]:

$$E[\xi_1 \xi_2 ... \xi_N] = \sum_P \prod_{j=1}^N E[\xi_j, \xi_{Pj}], \tag{6}$$

where the sum runs over *all* permutations P: $j \to Pj$ of the positive integers j. Specifically,

$$\begin{aligned} E[B_h^H B_i^H B_j^H B_k^H] = E[B_h^H B_i^H] E[B_j^H B_k^H] + E[B_h^H B_j^H] E[B_i^H B_k^H] \\ + E[B_h^H B_k^H] E[B_i^H B_j^H]. \end{aligned} \tag{7}$$

2.3 The power spectra of fGn and fBm

Eq. 5 leads to one of the defining properties of fGn: power spectrum with a singularity

$$S(f) = \sum_{k=-\infty}^{\infty} \rho(k) e^{-2\pi i f k},$$

which has an inverse transformation:

$$\int_{-1/2}^{1/2} S(f) e^{2\pi i f k} df = \rho(k).$$

Noting the equality

$$\sum_{k=-\infty}^{\infty} e^{-2\pi i k f} = \sum_{k=-\infty}^{\infty} \delta(f-k)$$

which accounts for the alias phenomenon [39], we have for any positive integer N and $f \in [-1/2, 1/2]$:

$$\begin{aligned} N^{2H} S(f) &= \sum_{\ell=-(N-1)}^{N-1} (N - |\ell|) \sum_{k=[f-N/2]+1}^{[f+N/2]} \frac{1}{N} S\left(\frac{f-k}{N}\right) e^{2\pi i \ell (f-k)/N} \\ &= \sum_{k=[f-N/2]+1}^{[f+N/2]} \frac{1}{N} S\left(\frac{f-k}{N}\right) \sum_{\ell=-(N-1)}^{N-1} (N - |\ell|) e^{2\pi i \ell (f-k)/N} \\ &= \sum_{k=[f-N/2]+1}^{[f+N/2]} \frac{1}{N} S\left(\frac{f-k}{N}\right) \frac{1 - \cos(2\pi(f-k))}{1 - \cos(2\pi(f-k)/N)} \end{aligned}$$

which can be simplified into:

$$\sum_{k=[f-N/2]+1}^{[f+N/2]} \frac{S((f-k)/N)}{1-\cos(2\pi(f-k)/N)} = \frac{N^{2H+1}S(f)}{1-\cos(2\pi f)}. \tag{8}$$

Based on this iterative relation, it is easy to verify that [38]

$$S(f) = C(1-\cos(2\pi f)) \sum_{m=-\infty}^{\infty} \frac{1}{|f+m|^{2H+1}}$$

satisfies the Eq. 8 for any constant C.

3 The Continuous-Time Fractional Gaussian Noise and Fractional Brownian Motion

In the previous section we introduced the discrete-time fBm and its increment, fGn. We now introduce the continuous-time fBm and fGn (ctfGn). A ctfGn is a stationary stochastic process with the following defining property:

$$\frac{1}{T^H}\int_0^T X(t)dt \sim X(0) \qquad (T>0),$$

where '$\sim$' again means equality in probability distribution; $E[X(t)] = 0$ and $E[X^2(t)] = \sigma^2$. Parallel to the derivation for Eq. 3 we have

$$E\left[B_t^H B_\tau^H\right] = \frac{\sigma^2}{2}\left[t^{2H} - 2(\tau - t)^{2H} + \tau^{2H}\right], \tag{9}$$

where

$$B_t^H = \int_0^t X(\xi)d\xi \tag{10}$$

is a continuous-time fractional Brownian motion (ctfBm). From Eq. 10 the ctfGn can be formally written as $X(t) = dB_t^H/dt$. Therefore, the covariance of $X(t)$ can be obtained from Eq. 9 [27]:

$$\begin{aligned} E[X(t)X(\tau)] &= \frac{d^2}{dtd\tau}E\left[B_t^H B_\tau^H\right] = -\sigma^2\frac{d^2|t-\tau|^{2H}}{dtd\tau} \\ &= 2H(2H-1)\sigma^2|t-\tau|^{2H-2} - 2H\sigma^2|t-\tau|^{2H-1}\frac{d^2|t-\tau|}{dtd\tau} \\ &= 2H(2H-1)\sigma^2|t-\tau|^{2H-2} + 2H\sigma^2|t-\tau|^{2H-1}\delta(t-\tau), \end{aligned}$$

and since $X(t)$ is stationary, we have autocorrelation function for ctfGn:

$$\rho(\tau) = E[X(0)X(\tau)] = 2H(2H-1)\sigma^2|\tau|^{2H-2} + 2H\sigma^2|\tau|^{2H-1}\delta(\tau). \tag{11}$$

For $H = 0.5$ (Wiener white noise), the first term in Eq. 11 is zero and the second term is Dirac's $\delta(\tau)$. For $H > 0.5$, the second term is zero. The integration of $\rho(t)$ has:

$$\int_{-\infty}^{\infty} \rho(t)dt = \begin{cases} \infty & 0.5 < H < 1 \\ \sigma^2 & H = 0.5 \\ 0 & 0 < H < 0.5\ . \end{cases} \tag{12}$$

Note that for $H < 0.5$, $\rho(t)$ is negative for all $t \neq 0$. The integrations of both terms in Eq. 11 do not converge in the traditional sense. One obtains the zero by a cancelation of two ∞'s.

We now calculate the spectral density function for the ctfGn:

$$S(f) = \int_{-\infty}^{\infty} \rho(t) e^{-2\pi i f t} dt. \tag{13}$$

For $H > 0.5$, we have:

$$S(f) = 2H(2H-1)\sigma^2 C f^{1-2H}, \qquad (H > 0.5) \tag{14}$$

where the constant

$$C = \int_{-\infty}^{\infty} |\xi|^{2H-2} e^{-2\pi i \xi} d\xi.$$

Therefore ctfGn has a simple power-law spectral density function over all frequencies f. However, for discrete-time fGn this is not the case.

For anti-correlated fGn with $H < 0.5$, we note again that the Fourier transformtion of $\rho(t)$ does not exist in the traditional sense.

4 Estimations of Power Spectra and their Statistical Accuracy

In applications, practical measurements only encounter discrete-time fractal time series. For a finite realization of the $\{B_k^H\}$, $y_0, y_1, y_2, ..., y_N$, the power spectrum of the motion is defined as

$$S_m(f) = \sum_{h=1}^{N} \sum_{k=1}^{N} y_h y_k e^{-i2\pi(k-h)f} = 2 \sum_{k \geq h=1}^{N} y_h y_k \cos\left[2\pi(k-h)f\right] \tag{15}$$

where subscript 'm' stands for motion. If we treat the Eq. 15 as a statistical estimation, then

$$\begin{aligned} E[S_m(f)] &= 2 \sum_{k \geq h=1}^{N} E[B_h^H B_k^H] \cos\left[2\pi(k-h)f\right] \\ &= \sigma^2 \sum_{h \geq k=1}^{N} [h^{2H} - (k-h)^{2H} + k^{2H}] \cos\left[2\pi(k-h)f\right]. \end{aligned} \tag{16}$$

For large N, the summation in the Eq. 16 can be evaluated asymptotically using the Euler-Maclaurin summation formula [3].

Similarly, for a finite realization of the fGn $\{X_k\}$, $x_0, x_1, x_2, ..., x_N$, the power spectrum of the noise is estimated by:

$$S_n(f) = 2 \sum_{k \geq h=1}^{N} x_h x_k \cos[2\pi(k-h)f], \tag{17}$$

$$\begin{aligned} E[S_n(f)] &= \sigma^2 \sum_{k \geq h=1}^{N} [(k-h-1)^{2H} - 2(k-h)^{2H} + (k-h+1)^{2H}] \cos[2\pi(k-h)f] \\ &= \sigma^2 \sum_{\ell=0}^{N} (N-\ell)[(\ell-1)^{2H} - 2\ell^{2H} + (\ell+1)^{2H}] \cos(2\pi\ell f), \end{aligned}$$

and finally the variance in the estimated power spectrum $S_n(f)$:

$$\begin{aligned} E[S_n^2(f)] &= 4 \sum_{k \geq h=1}^{N} \sum_{n \geq m=1}^{N} E[X_h X_k X_m X_n] \cos[2\pi(k-h)f] \cos[2\pi(n-m)f] \\ &= 4 \sum_{\ell=0}^{N} \sum_{m=0}^{N} E[X_0 X_\ell X_0 X_m](N-\ell)(N-m) \cos(2\pi\ell f) \cos(2\pi m f), \\ Var[S_n(f)] &= E[S_n^2(f)] - E^2[S_n(f)] \\ &= 4 \sum_{\ell,m=0}^{N} \{E[X_0 X_\ell X_0 X_m] - E[X_0 X_\ell]E[X_0 X_m]\} \\ &\quad \times (N-\ell)(N-m) \cos(2\pi\ell f) \cos(2\pi m f) \\ &= 4 \sum_{\ell,m=0}^{N} \{E[X_0^2]E[X_\ell X_m] + E[X_0 X_m]E[X_0 X_\ell]\} \\ &\quad \times (N-\ell)(N-m) \cos(2\pi\ell f) \cos(2\pi m f). \end{aligned}$$

5 The Shape of Two-dimensional fBm

fBm can also be used to model the geometric properties of polymers. The application of the theory of Brownian motion to polymers is one of the great successes of applied stochastic processes. Treating polymers as random walks, P.J. Flory developed a quantitative theory of polymer structures [15], for which he was awarded the Nobel Prize in chemistry in 1974. The random walk model remains the theoretical foundation for studying synthetic polymers and biological macromolecules today [28–30].

fBm can also be used to model the geometric properties of polymers like proteins. In a poor solvent, the size of such molecules grows asymptotic as N^ν

where N is the number of units in the polymer, and $\nu < 1$. This is in contrast to a polymer in a good solvent, which has $\nu > 1$ [15]. We have recently obtained analytical results, asymptotically for large N, on the mean-square radius of gyration of a two-dimensional polymer modeled by fBm

$$\frac{\sigma^2 N^{2H}}{(2H+1)(2H+2)}, \tag{18}$$

with the asymmetricity in its shape

$$2 - \frac{1}{2(H+1)^2}\left[\frac{1}{2(1+H)^2} + \frac{2H+1}{4(4H+1)} - \frac{1}{4H+3} - \frac{\Gamma^2(2H+2)}{\Gamma(4H+4)}\right]^{-1}, \tag{19}$$

and in the spatial distributions of its kth unit [31]. The distribution is Gaussian, with variance:

$$2\sigma^2 N^{2H}\left[\frac{(1-k/N)^{2H+1} + (k/N)^{2H+1} - 1}{2H+1} + \frac{1}{2H+2}\right]. \tag{20}$$

Eq. 20 gives the mean-square distance between the kth unit and the center of the molecule. These results generalize the classic work on random coil polymers [12,35].

6 The Fractal Geometry of fBm and fGn

There are many fractal geometric characteristics associated with an fBm or a fractional Gaussian random field. For example, the fractal dimension of fBm's graph in $(E+d)$-dimensions is $2-H$ [43]. One can also obtain the fractal dimension of its sample paths in E-dimensional Euclidean space. Let us consider an N-step trajectory of a Gaussian process with zero mean and variance $\rho_n = \sigma^2 n^{2H}$ $(n = 1, 2, ..., N)$. This defines an fBm with Hurst coefficient H $(0 < H < 1)$. Hence, in an E-dimensional Euclidean space the probability of the fBm starting at the origin and at its nth step reaching an ϵ-ball in the neighborhood of $\mathbf{x}$ $(\mathbf{x} \in R^E)$ is:

$$\frac{\exp[-\mathbf{x}^2/(2\rho_n)]}{(2\pi\rho_n)^{E/2}} V_\epsilon, \tag{21}$$

where V_ϵ is the volume of the ϵ-ball. The N points have a radius of gyration $\sim N^H$, which statistically characterizes the average size of all the possible N-step trajectories. Conversely, for a given sphere of radius R, the average length of the trajectory within the sphere is $L \sim R^{1/H}$. Therefore, according to Mandelbrot's notion of a fractal set [23], the trajectory has a fractal dimension $d_f = \frac{d\ln(L)}{d\ln(R)} = 1/H$. The mathematically rigorous version of this result states that the Hausdorff dimension of an fBm is $\min(E, 1/H)$ [20,42].

Does an fBm eventually reach every part of the E-dimensional space? To answer this question, we note that if ϵ is sufficiently small, we can further assume that Eq. 21 is the probability of the fBm is reaching the ϵ-ball the first time since the probability of reaching it the second time is $\propto V_\epsilon^2$. We therefore have the probability of the fBm eventually, irrespective of n, reaching the neighborhood of $\mathbf{x}$

$$V_\epsilon \sum_{n=1}^{\infty} \frac{\exp[-\mathbf{x}^2/(2\rho_n)]}{(2\pi\rho_n)^{E/2}} \text{ which, asymptotically, } \sim V_\epsilon \sum_{n=1}^{\infty} \frac{1}{n^{HE}}. \tag{22}$$

For sufficiently small ϵ, this probability will be less than unity if $HE > 1$. Therefore, there is a finite probability that the fBm will not reach the neighborhood of $\mathbf{x}$. Note that $H = 1/d_f$; hence $d_f < E$ indicates that the fractal dimension of the trajectory is less than that of the Euclidean space. Conversely, if $d_f \geq E$, then with probability 1 the fBm will visit any small ϵ-ball centered at $\mathbf{x}$. In fact, it will visit it an infinite number of times.

This result has a strong intuitive physical interpretation: if the sample path of a correlated random walk has a fractal dimension d_f, it can fill the Euclid space with dimension $E \leq d_f$, but will not for $E > d_f$. For classic random walk with $H = 0.5$, the above result is the well-known Polya's theorem, which states that a random walker will visit every place in one and two dimensions, but not in three dimensions [14].

The mathematical investigation of fBm also provides insight into the problem of self-avoiding random walk (SAW) whose path cannot intersect itself in space. This is still a significant unsolved problem in the statistical physics of polymers. It is known that SAWs have asymptotic variance $\sim n^{6/(E+2)}$ where E is the dimension of the Euclidean space for the random walk [11]. Although fBm is not a faithful model for SAW, these two models agree asymptotically if $H = 3/(E+2)$. For $E = 2$ this corresponds to $H = \frac{3}{4}$. It will be interesting to find whether it is possible to use fBm as an approximated model for SAW. This entails defining a measure for the approximation and is intimately related to the mathematical concept of *intersectional local time*, in which the significance of $H = \frac{3}{4}$ has been specifically noted [34].

7 Nonlinear Block Transformation and Stability of Its Fixed Point

The theory of self-similar fBm developed by Gallavotti, Jona-Lasinio and Sinai [16,38] is both a theory of stochastic processes and a theory of nonlinear dynamical systems. The unifying theme of these two aspects is the distribution function and its transforming semigroup [21]. The nonlinear dynamical aspect of the renormalization group method has been actively investigated as singular perturbations of differential equations in recent years [10,13,26].

Consider an infinite-dimensional random vector $X_1, X_2, ..., X_k, ...$ The block transformation can be expressed either as a linear transformation for the joint distribution function $f_{\{X_k\}}(x_1, x_2, ..., x_k, ..)$, $\mathcal{P}_N$, called Frobenius-Perron operator [21],

$$\mathcal{P}_N[f_{\{X_k\}}](x_1, x_2, ..., x_k, ..) = \int ... \int \prod_{i=1}^{\infty} \prod_{j=1}^{N} dy_{ij} f_{\{X_k\}}(N^H x_1 - y_{11}, y_{11} - y_{12}, ...,$$

$$y_{1N-1} - y_{1N}, N^H x_2 - y_{21}, ..., y_{2N-1} - y_{2N}, ..., N^H x_k - y_{k1}, ...) \qquad (23)$$

or a nonlinear transformation $\mathcal{A}_N^*$ for the singlet distribution $f_X(x)$ [19,38]:

$$\mathcal{A}_N^*[f_X](S) \triangleq f_X\left(\mathcal{A}_N^{-1} S\right), \qquad S \subset \mathbb{R}, \qquad (24)$$

where $\mathcal{A}_N$ is given in Eq. 1. $\mathcal{A}_N^*$ can not be given explicitly except when all the X's in Eq. 1 are independent. In that case,

$$\mathcal{A}_N^*[f_X](x) = N^H \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f_X(N^H x - y_1) f_X(y_1 - y_2) \dots$$

$$f_X(y_{N-1} - y_N) dy_1 dy_2 \dots dy_N. \qquad (25)$$

In general, instead of $\mathcal{A}^*$ let us consider the nonlinear transformation for the characteristic function $\varphi(s)$ of the density function $f_X(x)$:

$$\mathcal{R}_N[\varphi](s) = \varphi^N\left(\frac{s}{N^H}\right) e^{-N^{-2H} s^2 \sum_{k=1}^{N}(N-k)\gamma_k}, \qquad (26)$$

where

$$\varphi(s) = \int_{-\infty}^{\infty} f_X(x) e^{-isx} dx, \qquad (27)$$

and γ_k characterizes the correlation between X's.

Solving the fixed point from $\mathcal{R}_N[\varphi^*] = \varphi^*$, we have $\varphi^*(s) = e^{-\alpha s^2}$ and $\forall N$

$$\alpha N^{1-2H} + N^{-2H} \sum_{k=1}^{N} \gamma_k = \alpha. \qquad (28)$$

From Eq. 28 we have

$$\gamma_k = \alpha\left[(k-1)^{2H} - 2k^{2H} + (k+1)^{2H}\right], \qquad (29)$$

which is precisely $E\left[X_0 X_k\right]$ given in Eq. 4.

A linear stability analysis can be carried out for the fixed point φ^*. Let

$$\varphi(s) = \varphi^*(s)(1 + \epsilon h(s)), \qquad (30)$$

where ϵ is sufficiently small, $h(s)$ is an arbitrary function with $h(0) = h'(0) = 0$. Then

$$\begin{aligned}\mathcal{R}_N[\varphi](s) - \varphi^* &= \varphi^*(s)\left(1 + \epsilon h\left(s/N^H\right)\right)^N - \varphi^*(s) \\ &= \epsilon N \phi^*(s) h\left(s/N^H\right) + O(\epsilon^2).\end{aligned}$$

Therefore the linear approximation near the fixed point

$$\mathcal{L}[h](s) \triangleq N\phi^*(s)h\left(s/N^H\right), \tag{31}$$

and

$$\lim_{m\to\infty} \mathcal{L}^m[h](s) = \lim_{m\to\infty} \left(N\varphi^*(s)\right)^m h\left(s/N^{mH}\right) \sim \frac{s^2}{2}h''(0)N^{(1-2H)m}. \tag{32}$$

Since $|\varphi^*(s)| \leq |\varphi^*(0)| = 1$, the φ^* is stable for $H \geq 0.5$.

8 Discussion

fBm is a well-developed mathematical model of strongly correlated stochastic processes. It exhibits a wide-range of power-law behavior with critical exponents. It is closely related to the better-known Levy processes [24]. A larger literature for the latter exists. Levy processes also have found applications in numerious science and engineering problems [41] including glass relaxation [25] and diffusion in turbulent fluids [36,37]. Statistical and numerical aspects of fBm, as well as fitting the theoretical models, both discrete and continuous, to experimental time series have also been extensively investigated. The readers are refered to several recent works [7,8,44] and the references cited within.

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