Homework # 5

1. Let $X$ be a Poisson random variable taking values $n = 0, 1, 2, \cdots$ with probability mass function (pmf)
   \[ p_n = \Pr\{X = n\} = \frac{\lambda^n}{n!} e^{-\lambda}. \]
   
   (a) Show that its variance is $\lambda$.
   (b) Let $X_1$ and $X_2$ be two Poisson random variables with expected values $\lambda_1$ and $\lambda_2$, respectively. Show that $X = X_1 + X_2$ also has a Poisson distribution, with expected value $\lambda_1 + \lambda_2$.

2. For an integer-valued random variable $X$ with distribution $p_n$, the following function
   \[ g_X(s) = \sum_{n} p_n s^n, \]
   is called the probability generating function (pgf) of $X$.
   
   (a) What is the pgf of Poisson random variable with expected value $\lambda$?
   (b) What are the values of $g'(1)$, $g''(1)$, where $g'(s)$ stands for $g$’s derivative with respect to $s$? Any probabilistic meaning for these quantities?
   (c) What is the pgf of binomial distribution with parameters $p$ and $N$?

3. First, read the posted note on the web on “General birth and death dynamics of a single population”.

   Let $p_n(t)$ be the probability mass function of the number of individuals in a single population with stochastic dynamics. If we assume the birth and death rates (not per capita!) being $u_k$ and $w_k$ when there is $k$ individuals in the population, then we have
   \[ \frac{dp_k(t)}{dt} = p_{k-1}u_{k-1} - (u_k + w_k)p_k + w_{k+1}p_{k+1}, \quad k = 0, 1, \cdots \]
   Clearly, the strict birth and death rates $u_0 = w_0 = 0$. However, we shall assume that $u_0 \neq 0$ due to a very small rate of immigration. Then the very frist equation above for $k = 0$ is
   \[ \frac{dp_0(t)}{dt} = -u_0 p_0 + w_1 p_1. \]

   (a) Write down the equation for $k = 1, 2, \cdots$. We are interested in the steady state probability distribution, that is $\frac{dp_k}{dt} = 0$ for $k = 0, 1, \cdots$. You will see that one can then solve $p_k^{ss}$ in term of $p_0^{ss}$, $p_2^{ss}$ in terms of $p_1^{ss}$, etc. Find the general expression for $p_k^{ss}$ in terms of only $p_0^{ss}$.

   (b) Show that $p_0^{ss}$ should satisfy the equation
   \[ p_0^{ss} = \left[ 1 + \sum_{k=1}^{\infty} \left( \prod_{j=0}^{k-1} \frac{u_j}{w_{j+1}} \right) \right]^{-1}. \]
Why?

(c) Find the maximum and minimum of the discrete distribution $p^*_k$, as a function of $k$. Note that the position $k^*$ of an extremum can only be determined up to $k^* \pm 1$. Explain your result in term of population biology.

Additional Problems for AMATH 523

4. Using the information in Prob. #2 above on probability generating function, show that in the limit of $N \to \infty$, $p \to 0$, and $Np = \lambda$, the pgf of the binomial random variable converges to the pgf of Poisson distribution with parameter $\lambda$. 