

## Homework # 2

1. Consider a population consisting of identical and independent individual organisms, each with an exponentially distributed time for giving “birth”, with rate  $\lambda$ , and going “death”, with rate  $\mu$ .

(i) Now when the population has exactly  $n$  individuals, what is the probability distribution for the waiting time to the next birth? What is the probability distribution for the waiting time to the next death? What is the probability distribution for the waiting time to the next birth or death event?

(ii) Let  $p_n(t)$  be the probability of having exactly  $n$  individuals in the population at time  $t$ :

$$\sum_{n=0}^{\infty} p_n(t) = 1.$$

What system of differential equation  $p_n(t)$  should satisfy?

(iii) The mean population at time  $t$  is defined as

$$\langle n \rangle(t) = \sum_{n=0}^{\infty} n p_n(t).$$

Based on the system of differential equations you obtained in (ii), show that

$$\frac{d}{dt} \langle n \rangle = (\lambda - \mu) \langle n \rangle.$$

2. (i) Obtain the general solution to the pair of linear, homogeneous, ordinary differential equations (ODEs) with constant coefficients:

$$\begin{aligned} \frac{dX}{dt} &= -X - 3Y, \\ \frac{dY}{dt} &= -2X - 2Y. \end{aligned} \tag{1}$$

(ii) Find the particular solution to the above equation with initial values  $X(0) = Y(0) = 3$ .

(iii) Plot the two lines in a graph:

$$\begin{aligned} -X - 3Y &= 0, \\ -2X - 2Y &= 0; \end{aligned}$$

and find their intersection. What is the meaning of these two lines, and the meaning of the intersection point, in the  $(X, Y)$  plane for the ODE?

(iv) What is meant by a *trajectory* of the above system? Describe the behaviour of the trajectories of the system as  $t \rightarrow +\infty$  and  $t \rightarrow -\infty$ .

(v) Suppose  $x(t)$  and  $y(t)$  are numbers of two competing species which have an equilibrium point  $(x_c, y_c)$  where  $x_c, y_c$  are both positive. If  $X = x - x_c$  and  $Y = y - y_c$  satisfy the above equation, roughly sketch the phase portrait in the first quadrant of the  $(x, y)$  plane. Discuss, in detail, the fate of population  $x$  and  $y$ .

3. Let us again consider a pair of ODEs:

$$\begin{aligned}\frac{dX(t)}{dt} &= aX + bY, \\ \frac{dY(t)}{dt} &= cX + dY.\end{aligned}$$

This time, try to eliminate the unknown function  $Y$  and obtain a 2nd order ODE for  $X(t)$  in the form of

$$\frac{d^2 X}{dt^2} + \alpha \frac{dX}{dt} + \beta X = 0. \quad (2)$$

Find out  $\alpha$  and  $\beta$  in terms of  $a, b, c, d$ . Discuss the significance of your finding.

4. A chemostat is a continuous culture device used for growing and studying bacteria. Nutrient is added at a constant rate, say  $S_0\lambda$  (Mass/Volume)(1/Time), to the growth chamber where living cells are stirred in the enriched media. The growth chamber is continually adjusted to keep a constant volume by removing bacteria-containing fluid at the flow rate  $\lambda$ . Let  $S(t)$  denote the concentration of nutrient in the growth chamber at time  $t$ , and let  $B(t)$  denote the concentration of bacteria. (concentration = number/volume)

One mathematical model of the chemostat accounts for the addition of nutrient and the washing out of bacterial cells:

$$\begin{aligned}\frac{dS}{dt} &= \lambda(S_0 - S) - \frac{VSB}{K + S} \\ \frac{dB}{dt} &= \frac{VSB}{K + S}Y - \lambda B\end{aligned}$$

where  $\lambda$  is the flow rate,  $V$  is the maximum uptake rate of the nutrient by per bacterial cell,  $K$  is the saturation constant of nutrient uptake and  $Y$  is the yield of cells per unit nutrient taken up.

(a) Show that if  $\lambda$  is too large, then the cell population washes out. Determine the critical  $\lambda$ .

(b) Determine the optimal steady flow for maximal bacteria output while keeping the cell population steady.