Homework # 1

Students in 523 are required to complete additional problems on the second page

Office hour will be on Wednesday 9am-10:30am

1. Solve the nonlinear ordinary differential equation (1.2) on P. 3 of the reading material (from J. D. Murray: *Mathematical Biology I: An Introduction*); show that Eq. (1.3) is the solution to the ODE. Show all the steps; simply verifying the solution given in (1.3) is not acceptable.

2. Gompertzian growth model assumes that

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \mu N \ln\left(\frac{K}{N}\right).$$

(a) Solve the mathematical model to obtain $N(t)$ as function of t.
(b) Compare this model with the logistic growth model in Problem 1.

3. A nonlinear ordinary differential equation (ODE)

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = f(N)$$

is called *autonomous* if the right-hand-side $f(N)$ is not an explicit function of time t.

(a) Show that for a single-specie population dynamics described by an autonomous ODE, it is not possible to have oscillatory, nor periodic, solution $N(t)$.

(b) Now, some people might say that $N(t) = 2+\sin(t)$ can determine a $f(N)$ for which this is a solution of the ODE, thus it seems to be a *counterexample*. Try to explain why this is not the case?

4. If the per capita birth rate of a population is given by $[r - a(N-b)^2]$ where r, a, and b are positive parameters, write down a population model equation of the form $dN/dt = f(N)$. Nondimensionalise the equation so that the dynamics depend on a single dimensionless parameter $k = b(a/r)^{1/2}$. If u is your nondimensional population, sketch $f(u)$ for $k > 1$ and $k < 1$ and discuss how the qualitative behaviour of the solution changes with k and the initial condition.

5. [Extra credit] A model for the spruce budworm population $u(t)$ is governed by

$$\frac{\mathrm{d}u}{\mathrm{d}t} = ru\left(1 - \frac{u}{q}\right) - \frac{u^2}{1+u^2},$$

where r and q are positive dimensionless parameters. Let two variates function $x(r, p)$ be defined implicitly through the roots of the cubic equation

$$r - px - \frac{x}{1+x^2} = 0.$$

Either computationally using root(s) finding method, or using the cubic root formula, to plot the x as functions of p with several fixed values of r in a x versus p plot; as well as the x as functions of r with several different values of p in another plot.

Check out the online Wikipedia on Van der Waals equation,

`https://en.wikipedia.org/wiki/Van_der_Waals_equation`

compare your plots with the PV (P: pressure; V: volume) curves for different value of temperature T, the temperature. Also learn the concept of ***Maxwell's rule*** on the same webpage.

Discuss your finding.

Additional Problems for AMATH 523

6. The nonzero steady states to the Problem 5 are given by the intersection of the two curves

$$U(u) = r\left(1 - \frac{u}{q}\right), \quad V(u) = \frac{u}{1+u^2}.$$

Show, using the conditions for a double root, e.g., tangent between $U(u)$ and $V(u)$, that the curve in r, q (parameter) space which divides it into regions where there are 1 or 3 positive steady states is given parametrically by

$$r = \frac{2a^3}{(1+a^2)^2}, \quad q = \frac{2a^3}{a^2-1}. \tag{1}$$

Show that the two curves meet at a cusp, that is, where $dr/da = dq/da = 0$, at $a = \sqrt{3}$. Sketch the curves in r, q (parameter) space noting the limiting behaviour of $r(a)$ and $q(a)$ as $a \to \infty$ and $a \to 1$.

7. The predation $P(N)$ on a population $N(t)$ is very fast and a model for the prey population $N(t)$ satisfies

$$\frac{\mathrm{d}N}{\mathrm{d}t} = RN\left(1 - \frac{N}{K}\right) - P\left\{1 - \exp\left(-\frac{N^2}{\varepsilon A^2}\right)\right\}, \quad 0 < \varepsilon \ll 1,$$

where R, K, P and A are all positive constants. By an appropriate nondimensionalisation show that the equation is equivalent to

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = ru\left(1 - \frac{u}{q}\right) - \left[1 - \exp\left(-\frac{u^2}{\epsilon}\right)\right],$$

where r and q are positive parameters. Demonstrate that there are three possible nonzero steady states if r and q lie in a domain in r, q space given approximately by $rq > 4$. The notion of hysteresis is defined in the reading material; could this model exhibit hysteresis?