AMATH 383 Term Projects: Instructions and Resources

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Contents

Guidelines for Term Papers (Copyright ©by K. K. Tung)

COURSE PROJECTS

A major feature of this introductory mathematical modeling course is that students develop course projects and write term papers on those projects. These term papers are to be turned in on the last day of lectures, or electronically on or before that day. **No late submission is accepted.** The modeling projects and term papers may be done in groups of 1, 2, or 3 students. Please do not worry about this project excessively, but do start thinking about it from the very beginning. We know this is the first such experience for most of you.

The content of your project/term paper can be new and innovative research, or reviewing a few papers written by other scientists. A list of possible topics will be provied to you. You are strongly encouraged to pick one from the list, but this is not absolutely necessary. However, if you want to pick a topic on your own, you need the permission from the instructor ahead of time. The purpose of the project is that you learn to tackle a mathematical modeling problem with the following features:

(1) It should be a problem of interest to you.

(2) It should involve the mathematical techniques that you have studied in this class. It is also fine if it goes beyond what we have done in class and requires that you learn about some particular technique in greater depth.

(3) Since this course focuses on *mechanistic modeling*, it is not required to collect and/or analyzing any "real data". Comparing predictions from your mathematical model with real data would be nice, of course, but it is not essential. In any case, obtaining/collecting data itself should not be your main effort — that would be the task of a laboratory project.

(4) It is expected that you will have to use the library, physical or online, and identify relevent references in books and journals in order to do this project. Much useful information and data can also be found on the web. (But there is also a lot of nonsense out there. Remember that anyone can "publish" anything on the web and it is not subject to the same kind of editorial control as books or journals.)

The following are some pointers on how the proposal and term paper for this project should be structured. Please see the instructor or one of the TAs if you have additional questions.

TERM PAPERS

The lengths of term-project papers may vary, but we are expecting that you will need 10 or 12 pages of double-spaced text to describe a meaningful project. In addition, there may be figures, data, and/or computer programs. This means that we are talking about a total length of 15 to 20 pages.

The format of the paper will depend on what type of model you are looking at, but here are some key components that most papers should contain:

(1) ***Title and Abstract***

This is a short overview of the paper, a miniature version of 100 words or so. Someone reading the abstract should get a good idea of what problem has been tackled, what types of techniques were used to solve it, and what sort of solution was found. Most professional papers start with an abstract. It is very valuable for the potential reader, to help decide whether the paper is of interest and, if so, to get an overview of the whole picture before starting to read the details.

(2) ***Problem description***

Present the problem you are attempting to solve. Give some background. Explain why it is important or interesting. Outline the questions that you would like to answer.

(3) ***Simplifications***

You will probably need to simplify the problem in order to obtain a model that is appropriate for this project. Explain the ways in which you simplified the original problem and outline the assumptions that underlie these simplifications. Justify the assumptions, if possible, or discuss the limitations that are imposed on your model by your assumptions and simplifications.

(4) ***Mathematical model***

How did you turn the simplified problem into a mathematical model? Is there a standard mathematical paradigm that you are using, e.g., Newtonian mechanics, conservation of number of indivials, or linear programming? Or is it a problem of a different sort? How does it relate to standard problems? Define all of your variables, explain your notation, etc.

(5) ***Solution of the mathematical problem***

What techniques did you use to solve the mathematical problem? Were you able to use standard techniques, e.g., the simplex method for linear programming and linear analysis for differential equations? Did you need to develop a new analytical method and/or algorithm to solve the problem? Did you use a technique from the literature that we havent discussed in class? Explain in detail.

(6) ***Results and Discussion***

What were your results from solving the mathematical problem? How are these results interpretable in terms of the original problem? Are the results reasonable? If not, what are the possible failings of the model that led to poor results? If your model leads to a large problem that you cannot solve, try to formulate a smaller version that leads to reasonable results.

(7) ***Improvement***

How can you improve the model or solution technique so as to yield better results? How easy or diffcult is it to implement these improvements.

(8) ***Conclusions***

Summarize what you have done and what you have learned.

(9) ***References***

This is extremely important! Please include a bibliography if you have used any references, e.g., books, journal articles, web pages. Put a citation in the paper if you refer to a reference. An example of reference, Taubes (2001), is given below.

Please type the paper. You can use any word processing system that you like, and write in mathematical equations if necessary. Typesetting systems such as LaTeX are especially recommended.

I will keep the final copy of your project. Please be sure to xerox your final copy before turning it in.

References

Taubes, C.H. (2001) *Modeling Differential Equations in Biology.* Prentice-Hall, Upper Saddle River.

A List of Possible Term Projects (Copyright ©by Y.-A. Ma)

MECHANICAL SYSTEM

Levitron and tippe top

Mechanical systems with "three interacting bodies" in the full three dimensional space has 18 differential equations, three positions and three velocities for each point mass. Sometime one studies mechanical motion in two- or even one-dimensional system with external forces. J. E. Marsden discussed several four dimension mechanical systems. Two very interesting ones are the levitron and tippe top. For interesting videos of these systems, see https://www.youtube.com/watch?v=GMVtlNbMwHw for levitron; and https://www.youtube.com/watch?v=AyAgeUneFds for tippe top. Stability analysis for these two systems could be an interesting project. You want to find their similarities and differences. Examples can be found in Sec. 3A. of this reference: R. Krechetnikov and J. E. Marsden, Dissipation-induced instabilities in finite dimensions. *Reviews of Modern Physics*, **79**, 519–553 (2007).

Chaotic motion with energy conservation

You might think that the mechanical systems are simple. But that is very far from the the case. In fact, there is a vast amount of chaotic behaviors in those seemingly excessively deterministic systems. For example, the double pendulum system is a perfect example. For a nice video of it, see: https://www.youtube.com/watch?v=PrPYeu3GRLg. Starting from 1:28 of the video, you start to observe a chaotic behavior. However, if we don't consider frictional effect, the mechanical system has the law of energy conservative. How can you rationalize these competing considerations? Try to visualize the energy function, solve the trajectories on the energy surfaces, and explain why there can still be chaotic motion in the system.

Controling double pendulum

If you like engineering, there is a hard problem on controlling the double pendulum so that it reaches a specific position. If you can control the position and velocity of both rods, the question is how to do it fastest, with the constraint that the rods are not elastic. In reality, usually you can only control the lower rod. Then how to make both rods stand is already a not-so-easy problem. Try either of them (for the latter one, considering the friction effect actually makes your life easier).

Duffing oscillator

We have studied chaos in autonomous systems. But there are also systems that are not chaotic by themselves, but become chaotic when there is simple time-dependent input. The Duffing oscillator is such an example. It models a bouncing ball on a vibrating surface. Put a ball on your speaker and you will see it bouncing chaotically. You can first analyze the system's behavior when no input is present. Then for different input frequencies, analyze the different behaviors of the system.

ECOLOGICAL MODEL

We have discussed in class about the classical two species Lotka-Volterra model, in which there is a conserved quantity. Such models, however, are usually considered not realistic, and it is "structurally stable". That is, any perturbation to the equations would make the behavior of the system change drastically. There are many generalizations to the model, such as the parametric generalization considered in this reference: T. Ying, R. Yuan and Y. Ma, Dynamical behaviors determined by the Lyapunov function in competitive Lotka-Volterra systems. *Physical Review E* **87**, 012708 (2013). In the models with three species, there are more interesting behaviors.

Predator-prey model with structural stability

One possible project is to find a more complete model, in which the additional terms (or variation of the parameters) have ecological meaning and makes the system structurally stable.

Stochastic predator-prey model

There is discussion on how to formulate the Lotka-Volterra ecological model to finite population size in which stochasticity becomes important. See reference Y.-A. Ma and H. Qian, A thermodynamic theory of ecology: Helmholtz theorem for Lotka-Volterra equation, extended conservation law, and stochastic predator-prey dynamics. *Proceedings of the Royal Society A: Math. Phys. Engr.* **471**, 20150456 (2015). Under that consideration, the model becomes stochastic, while the conserved quantity in the original Lotka-Volterra model is the stationary distribution in the new stochastic model. It would be interesting to see the corresponding finite population size model for the Lotka-Volterra systems with general parameters, where the original deterministic dynamics is already "structurally stable".

NEUROSCIENCE

Within biology, neuroscience is an area that is currently highly active and rapidly growing with mathematical modeling in terms of dynamical systems approach. A great deal of new models come from different scales of the description of neural phenomena. On the cellular scale, many models concern with the spikes that nerons generate and pass to other nerons. This type of works were originated from the celebrated Hodgkin-Huxley (HH) theory that describes the membrane electrical voltage of a neuron in terms of voltage-gated ion channels and ionic currents, predicting the spiking behavior. You can find a nice explanation of this here: http://www.bem.fi/book/04/04.htm.

A high dimensional Hodgkin-Huxley model

One project that can originate from the Hodgkin-Huxley model is: study the behavior of multiple neurons in connection with each other, a higher dimensional Hodgkin-Huxley model. More specific questions such as this can be addressed: When would rhythms (periodic impulses) be generated; when would the neurons synchronize and desynchronize. N. Kopell's group has done such study. See S. R. Jones, D. Pinto, T. Kaper, and N. Kopell, Alpha-frequency rhythms desynchronize over long cortical distances: A modeling study. *J. Comput. Neurosci.* **9**, 271–291 (2000).

Model reduction of multiple interconnected HH neurons

Another project that HH model inspires is that: Simplify the model, preferably to a smaller number of dimensions, while keeping the behaviors approximately the same. Try the two models with same inputs, and observe that they indeed behave similarly, with your model reducing calculation burden significantly. Similar efforts can be found in many previous works, for example: H. Wang, Y. Yu, S. Wang, J. Yu, Bifurcation analysis of a two-dimensional simplified Hodgkin-Huxley model exposed to external electric fields. *Neural Computing and Applications*, **24**, 37–44 (2014).

Passive neuronal oscillation

In lower forms of organisms, the neural models are usually different. A simple oscillator model is often used to represent the behavior of one or a set of neurons. See: C. Zhang, et al. Neural mechanism of optimal limb coordination in crustacean swimming. *Proceedings of the National Academy of Sciences USA*, **111**, 13840-13845 (2014), and the references therein. First, observe the difference in the two models. Then explain for this difference physiologically. An older reference: D. E. Goldman, Potential, impedance, and rectification in membranes. *The Journal of General Physiology* **27**, 37–60 (1943), might be helpful since it's a model for passive neurons (more common in lower organisms), rather than excitable ones. Last, empirically explain for the reason of this difference.

On a larger neural network scale, people usually consider the frequency of spikes as the signal, instead of the voltage across the cellular membrane. Therefore, manysimpler models are used. One classical model is the Hopfield neural network model. A popular view of it is that the number of "attractors" denotes the memory power of the network. John Hopfield wrote two very famous papers: J. J. Hopfield, Neural networks and physical systems with emergent collective computational abilities. *Proceedings of the National Academy of Sciences USA* **79**, 2554–2558 (1982); and J. J. Hopfield and D. W. Tank, Neural computation of decisions in optimization problems. *Biological Cybernetics*, **52**, 141–152 (1985).

Hopfield neural network and attractors

The behaviors of the Hopfield neural network is largely determined by the connection matrix T_{ij}. One project is to discuss the influence of the connection matrix T_{ij} on the long term dynamics of the whole system, specifically, to the number of attractors there is in the system.

Hopfield neural network with asymmetric connections

There is a crucial assumption in the original Hopfield network, $T_{ij} = T_{ji}$ is considered to be symmetric. But this is not quite realistic. In real neural networks, the connection is usually not symmetric: Signal often come from axon of a neuron and is pass onto the dendrite of another neuron. Discuss the difference if asymmetric connection is allowed in the Hopfield network. Especially, show how complicated dynamic behaviors can come into play. A useful reference could be: D. Kleinfeld, Sequential state generation by model neural networks. *Proceedings of the National Academy of Sciences USA*, **83**, 9469–9473 (1986).

Neural network and machine learning

Some people find that the Hopfield network is more suitable for the purpose of constructing artificial neural networks for machine learning. A simple example can be found at: `http://www.cs.ucla.edu/~rosen/161/notes/hopfield.html`. After you have trained the connection matrix of the Hopfield network, you can use it to do denoising or pattern recognition (to recognize hand writing, for example). This is for the people who like more engineering (computer science) flavored projects.

CLIMATE DYNAMICS

The Lorenz equation is a highly simplification of the climate models. We have investigated a little the Lorenz equation in the class, especially its sensitive dependence on initial conditions.

Lyapunov exponent and Lorenz equation

A quantitative characterization of the sensitive dependence on initial condition is called the Lyapunov exponent, which is an average index for the sensitivity of the system on the whole attractor. The method of calculation can be found in: N. V. Kuznetsov and G. A. Leonov, On stability by the first approximation for discrete systems, *International Conference on Physics and Control*, pp. 596–599 (2005). Calculate this factor for the Lorenz equation, and explain why the weather prediction only works up to about ten days (For this purpose, a specific set of parameter values must be assumed in the Lorenz equation.

Ensemble approach to Lorenz equation

A popular method in weather prediction nowadays is called the ensemble forecasting. That's essentially why we are often told the chance of precipitation. A very simple explanation with examples can be found in this reference: M. S. Tracton and E. Kalnay, Operational ensemble prediction at the national meteorological center: Practical aspects. *Wea. Forecasting*, **8**, 379–398 (1993). You can make this even simpler: Take a circle around the prescribed initial condition for the Lorenz system. Numerically solve the Lorenz system with a "ton" (technical term) of initial conditions inside the circle, up to time t (You can prescribe those conditions uniformly, corresponding to uniform probability distribution). Then those initial conditions are taken with the Lorenz system to other states, and the initial circle is squeezed and expanded, becoming other shapes. Calculate how probability distribution of those points and the volume of that set of points change with time t. And explain why the probability of the system being in the vicinity of any state is becoming less and less.

Phase volume expansion

In fact, the previous two projects are closely related: The more sensitive a system is with respect to its initial condition, the faster its volume in the phase space expands. Verify this numerically for the Lorenz equations with different parameter values. Note that the Lyapunov exponent is different for different parameters.

CHEMICAL OSCILLATION

Chemical oscillation is a very interesting phenomenon. It has a fascinating history. See video: https://www.youtube.com/watch?v=_gyzhvMLImg.

Briggs-Rauscher reaction

Model of for the reaction in the video can be found in: I. Lengyel, G. Rabai, and I. R. Epstein, Experimental and modeling study of oscillations in the chlorine dioxide-iodine-malonic acid reaction. *Journal of the American Chemical Society*, **112**, 9104–9110 (1990). Analyze the model, and you will discover that in chemical reaction systems, the fixed point is actually not a spiral (real eigenvalues at equilibrium). In other words, the oscillation is only a transient behavior. To analyze that oscillating behavior, you can use auto-correlation function of the trajectories or other methods. This reference might also be useful to you: Y. Li, H. Qian and Y. Yi, Oscillations and multiscale dynamics in a closed chemical reaction system: Second law of thermodynamics and temporal complexity. *Journal of Chemical Physics*, **129**, 154505 (2008).