CHAPTER 9

Chaotic Dynamics of Lorenz Equation

The celestial mechanics problem posed by Magbus Gösta Mittag-Liffler in 1885, as a part of mathematical contest set up by Oscar II, King of Sweden and Norway, is as follows:

> Given a system of arbitrary many mass points that attract each other according to Newton's laws, assuming that no two points ever collide, give the coordinates of the individual points for all time as the sum of a uniformly convergent series whose terms are made up of known functions.

In response to this question, Henri Poincaré discovered that "It may happen that small differences in the initial positions may lead to enormous differences in the final phenomena. Prediction becomes impossible."

What are "known functions"? $\sin t$ and e^{-t} are certainly two well known ones, as well as $a_3t^3 + a_2t^2 + a_1t + a_0$. But what about $\Gamma(t)$? Checking out online, it is defined as

$$\Gamma(t) = \int_0^{\infty} x^{t-1}e^{-x}\mathrm{d}x. \tag{9.1}$$

In fact, it is easy to verify that for an integer value $t = n$, $\Gamma(n) = (n-1)!$. It can also be written as a product of infinite terms of known functions:

$$\Gamma(t) = \frac{1}{t}\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^t}{1+\frac{t}{n}}.$$

We now introduce Lorenz equation:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \sigma\big(-x+y\big), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= rx - y - xz, \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= xy - bz, \end{aligned} \tag{9.2}$$

in which $\sigma, r, b > 0$. This is a highly simplied model for rotating fluid with convection: $x(t)$ represents the rotational speed of the fluid, $y(t)$ represents the temperature difference, $z(t)$ characterizes the deviation from linear convection profile in the vertical direction.

9.0.1 Essential properties

We see that if one changes $(x, y) \to (-x, -y)$, the equation is unchanged. Therefore, the differential equation has a symmetry: If (x, y, z) is a solution, so is $(-x, -y, z)$. This means a solution is either itself symmtric, or it has a mirror image twin.

We also note that the divergence of the right-hand-side vector field is strictly negative for the entire $\mathbb{R}^3$:

$$\frac{\partial}{\partial x}\Big(\sigma(y-x)\Big) + \frac{\partial}{\partial y}\Big(rx - y - xz\Big) + \frac{\partial}{\partial z}\Big(xy - bz\Big) = -\sigma - 1 - b < 0. \quad (9.3)$$

This property is known as *volume contraction*.

9.0.2 Fixed points and their lossing stability with increasing r

There are three fixed points: the origin $(0, 0, 0)$, $C^+ = \big(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1\big)$, and its mirror image $C^- = \big(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1\big)$. Therefore, for $r < 1$, there is only the origin, which is stable: Linear analysis yields

$$\begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}_{(0,0,0)} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}, \quad (9.4)$$

which has the eigenvalue in z-direction being $-b$, and in the xy plan, the trace is $(-\sigma - 1)$ and the determinant is $\sigma(1-r)$. Therefore the origin changes from a stable node for $r < 1$ to a unstable saddle when $r > 1$.

In fact, for $r < 1$, a Lyapunov function $L(x, y, z) = \frac{1}{2}\big(x^2/\sigma + y^2 + z^2\big)$ can be used to prove the global attractiveness of the stable fixed point at the origin:

$$\frac{\mathrm{d}}{\mathrm{d}t} L\big[x(t), y(t), z(t)\big] = (1+r)xy - x^2 - y^2 - bz^2 \leq 0.$$

9.0.3 Hopf bifurcation

When $r > 1$, the origin is no longer stable. At the same time, a pair of new fixed points, C^+ and C^-, appear. Carrying out linear stability analysis, we obtain:

$$\begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}_{C^+} = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\beta \\ \beta & \beta & -b \end{pmatrix}, \quad (9.5)$$

in which $\beta = \sqrt{b(r-1)}$. The characteristic equtaion is

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2b\sigma(r-1) = 0. \quad (9.6)$$

It has one real and two complex eigenvalues. More importantly, when the real part of the two complex conjugate eigenvalues crossing zero, the two eigenvalues are purely imaginary $\pm i\omega$. This means

$$-i\omega^3 - (\sigma + b + 1)\omega^2 + i(r + \sigma)b\omega + 2b\sigma(r - 1) = 0, \tag{9.7}$$

which implies *two* equations, both have to be true at the Hopf bifurcation point:

$$\omega^2 = (r + \sigma)b, \ \text{ and } \ (\sigma + b + 1)\omega^2 - 2b\sigma(r - 1) = 0.$$

Therefore, we have

$$(r + \sigma)b = \frac{2b\sigma(r - 1)}{\sigma + b + 1}. \tag{9.8}$$

This yields a critical r_c for Hopf bifurcation

$$r_c = \sigma \left(\frac{\sigma + b + 3}{\sigma - b - 1} \right). \tag{9.9}$$

When $1 < r < r_c$, both C^+ and C^- are stable spirals; when $r > r_c$, they become unstable spirals. Then according to Hopf bifurcation theorem, there is a limit cycle near C^+ when r is near r_c. It turns out, however, the limit cycle is subcritical, unstable.

When $1 < r < r_c$, we note that the origin, being a saddle in $\mathbb{R}^3$ with a 2-dimensional stable manifold, divide the space into two parts.

The two unstable limit cycles, "divide" the state space $\mathbb{R}^3$ into three parts:

The motion is now in a strange state: It is repelled from one unstable object after another; it is confined to a set with zero volume; and yet it continues to move in this set without intersecting itself. This set is called a "strange attractor".

Symmetry breaking.

9.1 Poincaré section map

Geometric thinking and dynamical systems.

The trajectory is now moving around C^+ to C^- and back to C^+.

9.2 Geometric Lorenz attractor

9.3 Three different levels of erratic motions with complex dynamics

9.3.1 Ergodic motion

9.3.2 Mixing

9.3.3 Exact