CHAPTER 4

From Kepler's Laws to Newton's Universal Gravity: Data Science in an Ancient Time

Kepler's three laws states

I. A planet moves in an elliptical orbit with the Sun at one of its focus;

II. The area swept by the radius equal with equal time;

III. For different planets around the Sun, the squares of the times of full orbits (periods) are proportional to the cubes of the major axes.

4.1 Circular motion with centripetal force

A circle is a special case of an ellipse. If we replace the elliptical orbit in Kepler's laws by circular orbit, then we see the circular motion has to have constant tangential speed according to the statement II. Recall from high school physics, we had

> "The magnitude of the centripetal foce on an object of mass m moving at tangential speed v along a circular path with radius r is $F_c = mv^2/r$."

The period of the circular motion, thus, is

$$T = \frac{2\pi r}{v} \implies T^2 = \frac{4\pi^2 r^2 m}{rF_c} = \frac{4\pi^2 mr}{F_c}. \tag{4.1}$$

Therefore, the statement III that $T^2 \propto r^3$ implies $F_c \propto r^{-2}$, and vice versa!

4.2 The ellipse

To understand these three statement, we need to know the geometric object ellipse. In a Cartesian coordinate plane, the equation for an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{4.2}$$

in which a and b are called major or minor axes.

There are two foci F_1 and F_2 on the major axis, which are $2c$ apart. The point P on an ellipse has the property of $|F_1P| + |F_2P| = 2a$. Therefore, one has $b^2 + c^2 = a^2$. Using this defining property, one can derive the equation in (4.2):

$$\sqrt{(x-c)^2+y^2} + \sqrt{(x+c)^2+y^2} = 2a,$$
$$(x-c)^2 + y^2 = 4a^2 + (x+c)^2 + y^2 - 4a\sqrt{(x+c)^2+y^2},$$
$$\left(1 - \left(\frac{c}{a}\right)^2\right) x^2 + y^2 = a^2 - c^2,$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

In the polar coordinates centered at one of the foci, $x = c + r\cos\theta$ and $y = r\sin\theta$. Therefore, substituting these into (4.2) we have

$$\frac{(c + r\cos\theta)^2}{a^2} + \frac{(r\sin\theta)^2}{b^2} = 1,$$
$$b^2(c^2 + 2cr\cos\theta + r^2\cos^2\theta) + a^2r^2\sin^2\theta = a^2b^2,$$
$$a^2r^2 = \big(cr\cos\theta\big)^2 - 2b^2cr\cos\theta + b^4 = \Big(cr\cos\theta - b^2\Big)^2,$$
$$ar = b^2 - cr\cos\theta,$$
$$r = \frac{b^2}{a + c\cos\theta} = \frac{p}{1 + e\cos\theta},$$

in which $e = \frac{c}{a}$ is called eccentricity, $p = b^2/a = a(1 - e^2)$.

4.3 Calculus of a moving point with polar coordinates

To understand the three statements from Kepler, we develop the calculus in the polar coordinate system:

$$\vec{r} = r \cdot \hat{r}, \quad \vec{v} = \frac{\mathrm{d}r}{\mathrm{d}t}\hat{r} + r\frac{\mathrm{d}\hat{r}}{\mathrm{d}t}, \tag{4.3}$$

where

$$\begin{aligned} \frac{\mathrm{d}\hat{r}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}\Big) \\ &= \Big(-\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}\Big)\frac{\mathrm{d}\theta}{\mathrm{d}t} \\ &= \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\hat{\theta}(t), \end{aligned} \tag{4.4}$$

where $\hat{\theta} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}$, and

$$\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t} = -\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\hat{r}. \tag{4.5}$$

Therefore, one has velocity vector and acceleration vector in polar coordinate system:

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}, \tag{4.6a}$$

$$\vec{a} = \ddot{r}\hat{r} + 2\dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}. \tag{4.6b}$$

Or,

$$a_r = \ddot{r} - r\dot{\theta}^2 \text{ and } a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}. \tag{4.7}$$

4.4 Area under a curve in a polar coordinate system

The area

$$\Delta A = \frac{1}{2} r^2 \Delta\theta. \tag{4.8}$$

Therefore,

$$\frac{\Delta A}{\Delta t} = \frac{1}{2} r^2 \dot{\theta}. \tag{4.9}$$

The statement II in Kepler's laws implies $r^2\dot{\theta} = \text{constant}$, or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = 0 \implies 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = r \times a_\theta = 0$$
$$\implies \boxed{a_\theta = 0}. \tag{4.10}$$

Expressing an ellipse in a polar coordinate system:

$$r(\theta) = \frac{p}{1 + e\cos\theta}, \tag{4.11}$$

in which $p = a(1 - e^2)$, and e is called and eccentricity, which are related to the major and minor axes a and b:

$$e^2 = \frac{a^2 - b^2}{a^2}, \tag{4.12}$$

or

$$a = \frac{p}{1 - e^2}, \quad b = a\sqrt{1 - e^2}, \tag{4.13}$$

and the area within an ellipse is

$$\pi ab = \frac{\pi p^2}{(1 - e^2)^{3/2}}. \tag{4.14}$$

In a Cartesian coordinate system, of course:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{4.15}$$

4.5 Acceleration in radial direction

We therefore have, for a point mass moving on an ellipse

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} r\big(\theta(t)\big) &= \frac{\mathrm{d}r}{\mathrm{d}\theta}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) = \left(\frac{ep\sin\theta}{(1 + e\cos\theta)^2}\right)\dot{\theta} \\ &= \left(\frac{e\sin\theta}{p}\right) r^2\dot{\theta} = \frac{eh}{p}\sin\theta,\end{aligned} \tag{4.16}$$

where $h = r^2\dot{\theta}$ is a constant according to the statement II. Then,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} r\big(\theta(t)\big) = \frac{eh}{p}\big(\cos\theta\big)\dot{\theta} = \frac{eh}{p}\big(\cos\theta\big)\frac{h}{r^2} = \frac{eh^2}{pr^2}\cos\theta,$$

and noting $r^2\dot{\theta} = h$,

$$a_r = \ddot{r} - r\dot{\theta}^2 = \frac{eh^2}{pr^2}\cos\theta - \frac{h^2}{r^3} = \frac{h^2}{r^2}\left[\frac{e\cos\theta}{p} - \frac{1}{r}\right] = \frac{h^2}{r^2}\left[\frac{e\cos\theta}{p} - \frac{1+e\cos\theta}{p}\right]$$

$$\Longrightarrow \boxed{a_r = -\left(\frac{h^2}{p}\right)\frac{1}{r^2}}, \tag{4.17}$$

which shows an inverse square dependence on r! Note that the values of h and p are different for different planets.

4.6 The period of an elliptical orbit

The statement III in Kepler's laws says that T^2/a^3 = constant, independnet of the nature of the planets around the sun, where T is the period and a is the major axis of an orbit. Let us now compute the period of an elliptical orbit. Since

$$r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) = h, \text{ one has } \mathrm{d}t = \frac{r^2\mathrm{d}\theta}{h}. \tag{4.18}$$

Therefore,

$$T = \frac{1}{h}\int_0^{2\pi} r^2(\theta)\mathrm{d}\theta = \frac{2}{h}\Big(\pi ab\Big) = \frac{2\pi p^2}{h(1-e^2)^{3/2}}. \tag{4.19}$$

Since $a = \frac{p}{1-e^2}$,

$$T^2 = \frac{4\pi^2 p^4}{h^2(1-e^2)^3}, \text{ and } \frac{T^2}{a^3} = \frac{4\pi^2 p}{h^2}. \tag{4.20}$$

Kepler's third law states that $h^2/p = K$ is a constant independent of the planets that is moving around the sun.

Therefore, the acceleration in the radial direction, Eq. 4.17, becomes

$$a_r = -\frac{K}{r^2}. \tag{4.21}$$

4.7 Deduction of elliptical orbit based on Newtonian mechanics and universal gravity

Applying Newton's equation of motion, $ma = F$, to a planet treated as a point mass in a polar coordinate system, moving around the sun, we have

$$ma_\theta = 0 \text{ and } ma_r = -\frac{GMm}{r^2}, \tag{4.22}$$

where m and M are the masses of a planet and the sun, and G is the gravitational constant. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = ra_\theta,$$

$a_\theta = 0$ implies $r^2\dot{\theta} = h$ is a constant. Then the second equation in (4.22

$$a_r = \ddot{r} - r\dot{\theta}^2 = \ddot{r} - \frac{h^2}{r^3} = -\frac{K}{r^2}, \quad K = GM,$$

$$\ddot{r} = \frac{h^2}{r^3} - \frac{K}{r^2}. \tag{4.23}$$

On the other hand, since $r^2\dot{\theta} = h$,

$$\begin{aligned} \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{h}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}\right) = -h\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{r}\right)\right] \\ &= -h\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right)\dot{\theta} = -\frac{h^2}{r^2}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right). \end{aligned} \tag{4.24}$$

Therefore, the differential equation in (4.23) with respect to t becomes the differential equation with respect to θ in (4.25) below:

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right) = \frac{K}{h^2} - \frac{1}{r}, \tag{4.25}$$

or if denoting $x = 1/r$:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\theta^2} + x = \frac{K}{h^2}, \tag{4.26}$$

whose solution is

$$r(\theta) = \frac{1}{x(\theta)} = \frac{h^2}{K + A\cos\left(\theta - \theta_0\right)}. \tag{4.27}$$