$$\begin{bmatrix} 1-\mu_0 & 0 & 0 & \cdots & 0 \\ \mu_0 & 1-\mu_1 & 0 & \cdots & 0 \\ 0 & \mu_1 & 1-\mu_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mu_{K-1} & 1-\mu_K \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ \vdots \\ X_{K-1} \end{bmatrix} (n) + \begin{bmatrix} N_0 \\ N_1 \\ N_2 \\ \vdots \\ N_K \end{bmatrix},$$

in which X_ℓ is the number of students in grade ℓ, μ_ℓ is the percentage of the students in grade ℓ with satisfactory performance, and the inhomogeneous term N_ℓ represents the new students entering grade ℓ from outside. More compactly one writes:

$$X_\ell(n+1) = \big(1-\mu_\ell\big)X_\ell(n) + \mu_{\ell-1}X_{\ell-1}(n) + N_\ell, \tag{3.13}$$

$0 \le \ell \le K$.

The total student population,

$$X_T(n) = \sum_{\ell=0}^{K} X_\ell(n), \tag{3.14}$$

satisfies

$$X_T(n+1) = X_T(n) + \sum_{\ell=0}^{K} N_\ell - \mu_K X_K(n). \tag{3.15}$$

3.4 Networks and scaling concept (Ch. 2)

We shall now develop a mechanistic model for the "degree distribution" of a social network. As we shall see, the mathematics is nearly identical to that in the previous sections. But first let us introduce the notion of a graph, a mathematical object that describes a "network", which has nodes and edges. For a graph of n nodes, the maximal number of edges is $\frac{n(n-1)}{2}$. We call a node with k edges attached having a "degree k". Such a graph can be mathematically represented by a square matrix called "incidence matrix" which tallies all the nodes and connections: $G = \{g_{ij} | g_{ij} = 1$ if there is an edge between i and $j; g_{ij} = 0$ if there is no connection$\}$. It is clear that $g_{ij} = g_{ji}$, G is a symmetric matrix. We shall set the diagonal elements to zero.

3.4.1 Network with perferential attachment

Dynamic models like that in Eqs. (3.2) and (3.13 have wide applications beyond just population of biological species. For the graph, one classifies different nodes according to their degrees and then counts the number of nodes in each group. Let us now consider an problem from *network science*, discussed in Chapter 2. The so called ***preferential attachment model***. Let us use the language of the citation network, with $N_k(n)$ be the number of papers, among a "pool of total n papers", that are cited k

times per paper. So

$$\sum_{k=0}^{n} N_k(n) = n. \tag{3.16}$$

Furthermore, the total number of citation among the n papers is

$$\sum_{k=0}^{n} kN_k(n) = nm, \ \text{ where } \ m = \frac{1}{n}\sum_{k=1}^{n} kN_k(n) \tag{3.17}$$

is the the mean number of citations of a paper.

Now when a "new paper" is added into the pool, we have a distribution $N_k(n+1)$, which is related to the $N_k(n)$ in terms of the following mechanism:

i) The new paper cites a particular paper according to the popularity of that paper, determined by its current number of citations.

ii) The mean number of citation of the network m is a constant. We let there be total m citations from a new paper.

iii) Among the total nm citations existed, a particular paper in group N_k has k citations. So we shall let the probability of **each citation** from the new, $(n+1)^{th}$ paper citing a paper with already k citations be

$$\left(\frac{k+1}{n(m+1)}\right),$$

the additional 1 is to make sure a paper is at least has 1 citation to start with. Then from all m citations from the new, $(n+1)^{th}$ paper, the probability of citing a paper with already k citations will be

$$\mu_k(n) = \left(\frac{k+1}{n(m+1)}\right) m, \tag{3.18}$$

and the probability of the a paper with already k citations not being cited by the new one is $1 - \mu_k(n)$.

Let us recap: The extra 1 in the numerator and denominator takes care of the following situation: the probability of citing a paper with k citations is proportional to $k+1$, not k since we want paper with zero citation to have at least a non-zero probability to get started. The term in $(\cdots)$ is the probability of each citation from the new paper citing a paper in group N_k. The factor m takes care of the total m citations from the new paper. Note this is only approximately true when the probability $\frac{k+1}{n(m+1)}$ is very small. Note the μ_k in (3.18) is not only a function of k, but also inversely proportional to n; this is a new feature: It yields

$$\sum_{k=0}^{n} \mu_k(n) N_k(n) = \sum_{k=0}^{n} \left(\frac{(k+1)m}{n(m+1)}\right) N_k(n) = m, \tag{3.19}$$

as expected.

With the probability μ_k in (3.18), we have

$$\begin{cases} N_k(n+1) &=& \Big(1-\mu_k(n)\Big)N_k(n)+\mu_{k-1}(n)N_{k-1}(n), \\ & & (k\geq 2) \\ N_1(n+1) &=& \Big(1-\mu_1(n)\Big)N_1(n)+1. \end{cases} \tag{3.20}$$

We note that while $N_k(n)$ always changes with increasing n, the total number of papers, the ***proportion***

$$p_k(n)\equiv\frac{N_k(n)}{n} \tag{3.21}$$

actually will converge to a distribution that eventualy independent of increasing n. Following Eq. 3.20, the $p_k(n)$ satisfies

$$\begin{aligned} (n+1)p_k(n+1)-np_k(n) &= -\mu_k(n)np_k(n)+\mu_{k-1}(n)np_{k-1}(n) \\ &= -\frac{(k+1)m}{m+1}p_k(n)+\frac{km}{m+1}p_{k-1}(n). \end{aligned} \tag{3.22}$$

Let us denote

$$\lim_{n\to\infty}p_k(n)=p_k^*. \tag{3.23}$$

Then we obtain

$$p_k^*=-\frac{(k+1)m}{m+1}p_k^*+\frac{km}{m+1}p_{k-1}^*. \tag{3.24}$$

Therefore,

$$\frac{p_k^*}{p_{k-1}^*}=\frac{k}{k+2+1/m}=1-\frac{2+1/m}{k+2+1/m}. \tag{3.25}$$

For large k, $p_k^*/p_{k-1}^*\simeq 1-(2+1/m)/k$. This expression is actually consistent with a power law distribution $q_k\propto k^\lambda$ with large k:

$$\frac{q_k}{q_{k-1}}=\frac{k^\lambda}{(k-1)^\lambda}=\left(1-\frac{1}{k}\right)^{-\lambda}\simeq 1+\frac{\lambda}{k}. \tag{3.26}$$

We identify that the power $\lambda=-(2+1/m)$. Therefore, the degree distribution given in Eq. 3.25 is

$$p_k^*=p_1^*k^{-2-1/m}, \tag{3.27}$$

in which the p_1^* is determined from

$$\sum_{k=1}^\infty p_k^*=1 \implies p_1^*=\left(\sum_{k=1}^\infty k^{-2-1/m}\right)^{-1}. \tag{3.28}$$

3.4.2 Network with uniform attachment

The result in Eq. 3.27 should be compared and contrasted with the uniform attachment model, in which $\mu_k(n)=\frac{m}{n}$, independent of k. In this case, every paper gets an equal probability of being cited. Then Eq. 3.22 becomes

$$\begin{aligned}(n+1)p_k(n+1) - np_k(n) &= -\mu_k(n)np_k(n) + \mu_{k-1}(n)np_{k-1}(n) \\ &= -mp_k(n) + mp_{k-1}(n). \qquad (3.29)\end{aligned}$$

Thus, as $n \to \infty$, $p_k(n) \to p_k^*$, and

$$p_k^* = -mp_k^* + mp_{k-1}^*. \qquad (3.30)$$

That is,

$$\frac{p_k^*}{p_{k-1}^*} = \frac{m}{m+1}, \qquad (3.31)$$

which is independent of k. Thus,

$$p_k^* = p_0^* \left(\frac{m}{m+1}\right)^k, \;\; p_0^* = \frac{1}{m+1}. \qquad (3.32)$$

The p_0^* is determined by requireing the total probability

$$\sum_{k=0}^{\infty} p_k^* = 1 \quad \Longrightarrow \quad p_0^*\big(m+1\big) = 1.$$

The mean value then

$$\sum_{k=0}^{\infty} kp_k^* = \frac{\mu}{1-\mu} = m.$$

We see that this geometric form of $p_k^* = (1-\mu)\mu^k$, $\mu = \frac{m}{m+1}$, is very different from the power law distribution $\sim k^{-2-1/m}$. The former decays to zero as $k \to \infty$ much faster than the latter.

3.4.3 Degree distribution and incidence matrix

What is the mathematical relation between the incidence matrix $G = \{g_{ij}\}$ and the degree distribution of the graph? Actually, we have

$$N_k = \sum_{\ell=0}^{n} \delta \left(k, \sum_{j=1}^{n} g_{\ell j} \right), \qquad (3.33)$$

in which the δ-function is $\delta(i,j) = 1$ if $i = j$ and $\delta(i,j) = 0$ if $i \neq j$. The sum $\sum_{j=1}^{n} g_{\ell j}$ is the degree of the ℓ^{th} node. So we see that the incidence matrix is a much more informative description of a graph (network), the degree distribution is only one of the many aspects of the graph. This is again an example of mechanical "Lagrangian representation" to chemical "Eulerian representation": From resolving each and every node with an identity to simply counting the number of nodes according to a classification: the degree in this case.

Now, the dynamics of the evolving network can be represented with full details by the following n-dependent matrix

$$G(n) = \begin{bmatrix} \begin{bmatrix} & & \\ & G(n-1) & \\ & & \end{bmatrix} & \begin{matrix} \vdots \\ \vdots \\ \vdots \end{matrix} \\ \cdots \quad \cdots \quad \cdots & 0 \end{bmatrix} \tag{3.34}$$

in which $G(n)$ is a $n \times n$ symmetric matrix and $G(n+1)$ is a $(n+1) \times (n+1)$ matrix. When a new node is added, there is an additional row and similarly an additional column, which represents the attachements of the new, $(n+1)^{th}$ node to the other n existing nodes.

Not everythings can be represented in a graph. For example, a chemical reaction

$$A + B \rightleftharpoons C + D$$

suggests that there are two species at the end of an edge, if one uses a node to represent a chemical species. One can not represent a system of chemical reactions use a simple graph.