CHAPTER 3

Discrete-Time Population Dynamics, Networks and Scaling Concept

We have already encountered discrete-time population dynamics, in both Fibonacci sequence derived from a rabbit population dynamics and the mortgage payment calculation. The former deals with a *linear, second-order, constant coefficient, homogeneous difference equation*, and the latter involves a *linear, first-order, constant coefficient, inhomogeneous difference equation*. We have learned the standard methods for solving these two types of equations.

In this chapters, we are going to study more dynamics with discrete time: *e.g.*, a model of change in terms of a step after a step. Some models are nonlinear; others are systems of a large (even *denumerable*) number of equations.

3.1 A pharmacokinetic model for discrete medicine uptakes (Ch. 4)

Even a dynamics is in continuous time by nature, usually laboratory measurements and engineering manipulations are only performed in discrete time steps. We shall first consider one such exmample. For more discussion on simple model with continuous time, see Chapter 4 of the textbook.

Let $C(t)$ be the amount of medicine in the blood stream of a person. Let us further assume that the person take the medicine regularly with time intervals T and amount C_0, which starts at $t = 0$. Then for $t \in [0, T)$, one has the kinetic equation

$$\frac{\mathrm{d}C(t)}{\mathrm{d}t} = -kC(t), \;\; C(0) = C_0, \tag{3.1}$$

where k is the rate for drug clearance. The solution is

$$C(t) = C_0 e^{-kt}, \implies C(T^-) = C_0 e^{-kT},$$

where T^- is the moment before taking the medicine at time $t = T$, and T^+ below is the moment after taking the medicine:

$$C(T^+) = C_0 e^{-kT} + C_0.$$

Applying the above process again and again, we have

$$C\big((nT)^-\big) = C_0\Big(e^{-nkT} + \cdots + e^{-2kT} + e^{-kT}\Big),$$

$$C\big((nT)^+\big) = C_0\Big(e^{-nkT} + \cdots + e^{-2kT} + e^{-kT} + 1\Big).$$

The ℓ^{th} term in the sum represents the remaining drug in the blood stream from the ℓ^{th} uptake at time ℓT.

In the limit of $n \to \infty$, after a person having taking the drug forever, we have the concentration of the medicine pre- and post- regular uptaking:

$$\lim_{n\to\infty} C\big((nT)^-\big) = \frac{C_0 e^{-kT}}{1-e^{-kT}},$$
$$\lim_{n\to\infty} C\big((nT)^+\big) = \frac{C_0}{1-e^{-kT}}.$$

We note this model is remarkable similar to the mortagate problem, however with some crucial differences.

3.2 Population dynamics with age distribution

When the mathematical model for population dynamics is applying to a human population, more often than not the age of the individuals in a population matters, for example one's social behavior (in sociology), one's health conditions (in public health studies), or one's risk for a car accident (in insurance management), etc. In all these situations, one is interested in the populations of individuals in each and evey age group, and how this *distribution* changes with time.

Let $N_k(n)$ be the population size, e.g., the number of individuals, with age k in the n^{th} year. Then clearly without any birth and death, there is a "shift" of age from a year to the next year. Combining this with death in each sub-population with probability of death per capita per year d_k, one has a dynamic equation

$$N_k(n+1) = N_{k-1}(n) - d_k N_k(n), \tag{3.2}$$

$n \geq 0$, with the new borns which are considered as age 0:

$$N_0(n+1) = \sum_{\ell=1}^{\infty} b_\ell N_\ell(n), \tag{3.3}$$

where b_ℓ is per year number of births of each individual with age ℓ. We see that Eqs. (3.2) and (3.3) together can be expressed in terms of a *matrix equation*

$$\begin{bmatrix} N_0 \\ N_1 \\ N_2 \\ \vdots \\ N_K \end{bmatrix}(n+1) = \begin{bmatrix} 0 & b_1 & b_2 & \cdots & b_K \\ 1 & -d_1 & 0 & \cdots & 0 \\ 0 & 1 & -d_2 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 1 & -d_K \end{bmatrix} \begin{bmatrix} N_0 \\ N_1 \\ N_2 \\ \vdots \\ N_K \end{bmatrix}(n) \tag{3.4}$$

where we have assumed that the oldest age is K.

We note that the percentage of death per year, d_k, is related to the percentage survival per year $s_k = 1 - d_k$. Then another way to express the population dynamics with age distribution is

$$\begin{bmatrix} N_0 \\ N_1 \\ N_2 \\ \vdots \\ N_K \end{bmatrix}(n+1) = \begin{bmatrix} 0 & b_1 & b_2 & \cdots & b_K \\ s_0 & 0 & 0 & \cdots & 0 \\ 0 & s_1 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & s_{K-1} & 0 \end{bmatrix} \begin{bmatrix} N_0 \\ N_1 \\ N_2 \\ \vdots \\ N_K \end{bmatrix}(n). \qquad (3.5)$$

The matrix in (3.5) is known as a Leslie matrix in mathematical ecology.

3.2.1 Eigenvalues and their dependence on a model parameter

Let $\boldsymbol{A}$ denote the $(K+1) \times (K+1)$ matrix in Eq. 3.5, which can be then written as $\vec{N}(t+1) = \boldsymbol{A}\vec{N}(t)$. Therefore $\vec{N}(t) = \boldsymbol{A}^t\vec{N}(0)$, where $t = 0, 1, \cdots$.

From linear algebra, we know that a matrix like $\boldsymbol{A}$ has $K+1$ eigenvalues $\lambda_0 > \lambda_1 > \cdots > \lambda_K$, and corresponding to eigenvalue λ_ℓ, a left eigenvector

$$\mathbf{v}^{(\ell)} = \left(v_0^{(\ell)}, v_1^{(\ell)}, \cdots, v_K^{(\ell)}\right),$$

and a right eigenvector

$$\mathbf{w}^{(\ell)} = \begin{pmatrix} w_0^{(\ell)} \\ w_1^{(\ell)} \\ \vdots \\ w_K^{(\ell)} \end{pmatrix}.$$

The left eigenvectors and right eigenvectors satisfy an orthonormal relation

$$\mathbf{v}^{(\ell)} \cdot \mathbf{w}^{(m)} = \delta_{\ell m}, \qquad (3.6)$$

where $\delta_{\ell m} = 1$ when $\ell = m$, and $= 0$ when $\ell \neq m$. Then, $\boldsymbol{A}$ can be written as

$$\boldsymbol{A} = \begin{pmatrix} w_0^{(0)} & w_0^{(1)} & \cdots & w_0^{(K)} \\ w_1^{(0)} & w_1^{(1)} & \cdots & w_1^{(K)} \\ \vdots & \vdots & \cdots & \vdots \\ w_K^{(0)} & w_K^{(1)} & \cdots & w_K^{(K)} \end{pmatrix} \begin{pmatrix} \lambda_0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_K \end{pmatrix} \begin{pmatrix} v_0^{(0)} & v_1^{(0)} & \cdots & v_K^{(0)} \\ v_0^{(1)} & v_1^{(1)} & \cdots & v_K^{(1)} \\ \vdots & \vdots & \cdots & \vdots \\ v_0^{(K)} & v_1^{(K)} & \cdots & v_K^{(K)} \end{pmatrix}, \qquad (3.7)$$

and

$$\boldsymbol{A}^t = \begin{pmatrix} w_0^{(0)} & w_0^{(1)} & \cdots & w_0^{(K)} \\ w_1^{(0)} & w_1^{(1)} & \cdots & w_1^{(K)} \\ \vdots & \vdots & \cdots & \vdots \\ w_K^{(0)} & w_K^{(1)} & \cdots & w_K^{(K)} \end{pmatrix} \begin{pmatrix} \lambda_0^t & 0 & \cdots & 0 \\ 0 & \lambda_1^t & \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_K^t \end{pmatrix} \begin{pmatrix} v_0^{(0)} & v_1^{(0)} & \cdots & v_K^{(0)} \\ v_0^{(1)} & v_1^{(1)} & \cdots & v_K^{(1)} \\ \vdots & \vdots & \cdots & \vdots \\ v_0^{(K)} & v_1^{(K)} & \cdots & v_K^{(K)} \end{pmatrix}. \qquad (3.8)$$

Therefore, for a very large t, in "the long time limit", we have

$$\vec{N}(t) \simeq \alpha\lambda_0^t \begin{pmatrix} w_0^{(0)} \\ w_1^{(0)} \\ \vdots \\ w_K^{(0)} \end{pmatrix}, \text{ where } \alpha = \sum_{j=0}^{K} v_j^{(0)} N_j(0). \tag{3.9}$$

λ_0 is the annual growth rate of the population, and the corresponding eigenvector $\mathbf{w}^{(0)}$ gives the *stationary age distribution* within the long time population. According to Perron-Frobenius theorem, the elements of the $\mathbf{w}^{(0)}$ associated with the largest eigenvalue λ_0 are all non-negative when all entries of $\boldsymbol{A}$ are non-negative.

We now show a very important result on the sensitivity of the eigenvalue λ_ℓ with respect to a change in the entry a_{ij}, keeping all other entries fixed:

$$\frac{\partial \lambda_\ell}{\partial a_{ij}} = v_i^{(\ell)} w_j^{(\ell)}. \tag{3.10}$$

To prove this, we note

$$\lambda_\ell = \sum_{h,k} v_h^{(\ell)} a_{hk} w_k^{(\ell)} \text{ and } 1 = \sum_k v_k^{(\ell)} w_k^{(\ell)}. \tag{3.11}$$

Then,

$$\begin{aligned} \frac{\partial \lambda_\ell}{\partial a_{ij}} &= v_i^{(\ell)} w_j^{(\ell)} + \sum_{h,k} a_{hk} \left(\frac{\partial v_h^{(\ell)}}{\partial a_{ij}} w_k^{(\ell)} + v_h^{(\ell)} \frac{\partial w_k^{(\ell)}}{\partial a_{ij}} \right) \\ &= v_i^{(\ell)} w_j^{(\ell)} + \sum_h \frac{\partial v_h^{(\ell)}}{\partial a_{ij}} \lambda_\ell w_h^{(\ell)} + \sum_k \lambda_\ell v_k^{(\ell)} \frac{\partial w_k^{(\ell)}}{\partial a_{ij}} \\ &= v_i^{(\ell)} w_j^{(\ell)} + \lambda_\ell \frac{\partial}{\partial a_{ij}} \left(\sum_k v_k^{(\ell)} w_k^{(\ell)} \right) \\ &= v_i^{(\ell)} w_j^{(\ell)}. \end{aligned}$$

3.3 Population dynamics in a school district

While growing old year by year is obligatory, the grade distribution among the student population in a school district is not. In this case, a fraction (hopefully very small!) of the students will remaining in the same grade due to unacceptable performance, while others will move into a grade higher. In this case, we have

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ \vdots \\ X_K \end{bmatrix} (n+1) = \tag{3.12}$$

$$\begin{bmatrix} 1-\mu_0 & 0 & 0 & \cdots & 0 \\ \mu_0 & 1-\mu_1 & 0 & \cdots & 0 \\ 0 & \mu_1 & 1-\mu_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mu_{K-1} & 1-\mu_K \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ \vdots \\ X_{K-1} \end{bmatrix} (n) + \begin{bmatrix} N_0 \\ N_1 \\ N_2 \\ \vdots \\ N_K \end{bmatrix},$$

in which X_ℓ is the number of students in grade ℓ, μ_ℓ is the percentage of the students in grade ℓ with satisfactory performance, and the inhomogeneous term N_ℓ represents the new students entering grade ℓ from outside. More compactly one writes:

$$X_\ell(n+1) = \big(1-\mu_\ell\big)X_\ell(n) + \mu_{\ell-1}X_{\ell-1}(n) + N_\ell, \tag{3.13}$$

$0 \le \ell \le K$.

The total student population,

$$X_T(n) = \sum_{\ell=0}^{K} X_\ell(n), \tag{3.14}$$

satisfies

$$X_T(n+1) = X_T(n) + \sum_{\ell=0}^{K} N_\ell - \mu_K X_K(n). \tag{3.15}$$

3.4 Networks and scaling concept (Ch. 2)

We shall now develop a mechanistic model for the "degree distribution" of a social network. As we shall see, the mathematics is nearly identical to that in the previous sections. But first let us introduce the notion of a graph, a mathematical object that describes a "network", which has nodes and edges. For a graph of n nodes, the maximal number of edges is $\frac{n(n-1)}{2}$. We call a node with k edges attached having a "degree k". Such a graph can be mathematically represented by a square matrix called "incidence matrix" which tallies all the nodes and connections: $G = \{g_{ij} | g_{ij} = 1$ if there is an edge between i and j; $g_{ij} = 0$ if there is no connection$\}$. It is clear that $g_{ij} = g_{ji}$, G is a symmetric matrix. We shall set the diagonal elements to zero.

3.4.1 Network with perferential attachment

Dynamic models like that in Eqs. (3.2) and (3.13 have wide applications beyond just population of biological species. Let us now consider an problem from *network science*, discussed in Chapter 2. The so called ***preferential attachment model***. Let us use the language of the citation network, with $N_k(n)$ be the number of papers, among a "pool of total n papers", that are cited k times per paper. So

$$\sum_{k=0}^{n} N_k(n) = n. \tag{3.16}$$