AMATH 383 Homework Assignment #3
(Due in class: October 24)

1. Consider the dynamics of a population with n subpopulations:

$$\frac{\mathrm{d}X_i(t)}{\mathrm{d}t} = r_i X_i, \quad i = 1, 2, \cdots, n,$$

in which $X_i(t)$ is the population size of the ith subpopulation, constant r_i is its ***per capita growth rate***. Let x_i $(1 \le i \le n)$ present the *fraction* of the subpopulation X_i among the total population:

$$x_i = \frac{X_i}{\sum_{i=1}^n X_i}.$$

(a) Derive the corresponding system of differential equations for $x_i(t)$.

(b) Show that the per capita growth rate for the total population at time t is

$$\sum_{i=1}^n r_i x_i(t).$$

(c) Show that

$$\sum_{i=1}^n \left(\frac{\mathrm{d}x_i(t)}{\mathrm{d}t}\right) = 0,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^n r_i x_i(t)\right) \ge 0.$$

2. (a) Exercise 3 of Chapter 6, on page 109;

(b) Exercise 5 of Chapter 6, again on page 109.

Note that you do not have to read the Chapter 6 to do these problems.

3. In the Problem 1, let $n = 2$. Then you have the dynamics of a total population with two sub-populations X_1 and X_2:

$$\frac{\mathrm{d}X_1(t)}{\mathrm{d}t} = r_1 X_1, \quad \frac{\mathrm{d}X_2(t)}{\mathrm{d}t} = r_2 X_2.$$

(a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{X_1}{X_1 + X_2}\right) = \frac{(r_1 - r_2)X_1 X_2}{(X_1 + X_2)^2}.$$

Based on this mathematical expression, discuss what happens to the two sub-populations if $r_1 > r_2$, and if $r_2 > r_1$? What general conclusions can you reach for the total population?

(b) Now consider a biological population follows sexual reproduction. A simple population dynamic model for the two sexes X_1 and X_2 is

$$\frac{\mathrm{d}X_1(t)}{\mathrm{d}t} = r_1 X_1 X_2,$$
$$\frac{\mathrm{d}X_2(t)}{\mathrm{d}t} = r_2 X_1 X_2,$$

in which both $r_1, r_2 > 0$. Derive the expression for

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{X_1}{X_1 + X_2}\right).$$

Based on the mathematical expression, discuss what happens to the two sub-populations. What general conclusions can you reach for the total population?

(c) If exists, what is the steady population ratio, in the long-time limit, between X_1 and X_2?

4. ***The Hardy-Weinberg law in genetics.***

This law concerns the genetic make-up of a population from one generation to the next. It states that in sexually reproducing organisms, in the absence of genetic mutation, factors (called alleles) determining inherited traits are passed down unchanged from generation to generation. We want to show that the law is true.

A gene of an individal is made of a ***pair*** of alleles. Consider the simple case of only two possible alleles, A and B, in a gene. Let p_n be the probability of occurrence of the A allele in a population in generation n, and q_n be that of the B allele. $p_n + q_n = 1$ for all n. These two alleles combine to form three possible ***genotypes*** for the gene: AA, BB and AB, with probability p_n^2, q_n^2 and $2p_nq_n$, respectively.

The probability of the occurrence of the A alleles in the $n+1$ generation is denoted by p_{n+1} and that of the B alleles by q_{n+1}. We write

$$p_{n+1} = f(p_n, q_n), \quad q_{n+1} = g(p_n, q_n).$$

The alleles are passed from one generation to the next by ***random mating***: That is, for two individuals random picked from the n generation with genes XY and VW, their offspring has the gene made of one allele randomly chosen from XY and another one randomly chosen from VW.

Show that $f(p_n, q_n) = p_n$ and $g(p_n, q_n) = q_n$. Therefore

$$p_{n+1} = p_n \text{ and } q_{n+1} = q_n,$$

which are independent of n! This shows that the allele frequencies are passed down unchanged from generation to generation.

This simple mathematical result has very significant biological implications. You might find some more information on this problem in this paper, pp. 384–386.

`http://www.ams.org/notices/200803/200803-full-issue.pdf`

The paper is not needed for solving the problem, though.