AMATH 383 Homework Assignment #2  
(Due in class: Thu., April. 12)

1. If $T_1$ is a non-negative random time with probability density function $f_1(t)$, then

$$\int_0^{\infty} f_1(t)dt = 1.$$  

And its mean value (also called expected value) and variance can be obtained from

$$\langle T_1 \rangle = \int_0^{\infty} tf_1(t)dt,$$

$$\text{Var}[T_1] = \left( \int_0^{\infty} t^2 f_1(t)dt \right) - \langle T_1 \rangle^2.$$  

Now consider a second random time $T_2$ with probability density function $f_2(t)$. If $T_1$ and $T_2$ are statistically independent, then the probability density function for the random time $T_3 = T_1 + T_2$ is

$$f_3(t) = \int_0^{t} f_1(s) f_2(t - s)ds.$$  

(a) When $T_1$ is an exponentially distributed random time with mean time $\tau_1$, and $T_2$ is an exponentially distributed random time with mean time $\tau_2$, what is the distribution for $T_3$?

(b) Same $T_1$ and $T_2$ as in (a), what is the mean and the variance of $T_3$?

2. If random time $T \ (\geq 0)$ has a probability density function (pdf) $f_T(t)$, then

$$F_T(t) = \int_0^{t} f_T(s)ds$$

is called the cumulative distribution function (cdf) of $T$. It represents the probability for $T \leq t$: $F_T(t) = \Pr \{ T \leq t \}$. Show that if $T$ has an upper bound $M: T \leq M$, then

$$\langle T \rangle = \int_0^{\infty} [1 - F(t)]dt.$$  

3. Exercise 7 of Chapter 3.

4. Exercise 4 of Chapter 4.

5. Exercise 3 of Chapter 6, on page 109. Note that you do not have to read the Chapter 6 to do this problem. But you should have finished reading Chapters 3 and 4 by now.

6. Consider a problem in pharmacokinetics concerned with the dose-response relationship of a drug with its rate of clearance $\lambda C^2$, where $C(t)$ denotes the concentration of the drug in a patient’s body at time $t$. Let $C_0$ be the concentration at time $t = 0$, then

$$\frac{dC}{dt} = -\lambda C^2.$$  

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Now suppose constant doses $C_0$ are given at equal intervals of time $T$.

(a) Find the amount of drug in the body immediately after the $n^{th}$ dose, in terms of that of the $(n-1)^{th}$ dose.

(b) As $n$ tends to infinity, what is the steady state concentration of the drug in the body?

(c) Consider the model of a continuous intravenous injection with rate $I$:

$$\frac{dC}{dt} = -\lambda C^2 + I. \quad (2)$$

Solve the $C(t)$ and compare $C(\infty)$ with the result in (b).