BIOST 572 Introductory Talk

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Enjoy every sandwich

Warren Zevon (1947-2003)
American singer-songwriter

MODEL-ROBUST REGRESSION AND A BAYESIAN "SANDWICH" ESTIMATOR

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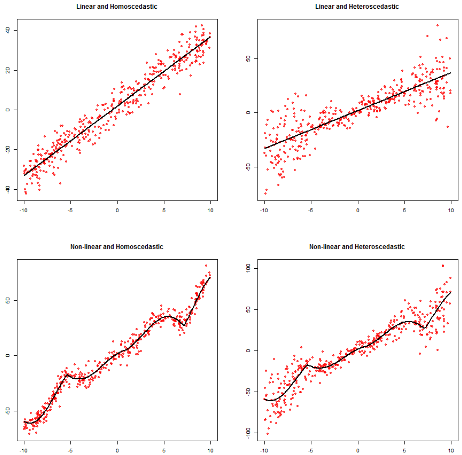
Classical Assumptions

- Outcome variable linearly related to covariates on average
- Random variations from the linear trend are homoscedastic
- Random variations are Normally distributed

Under these assumptions...

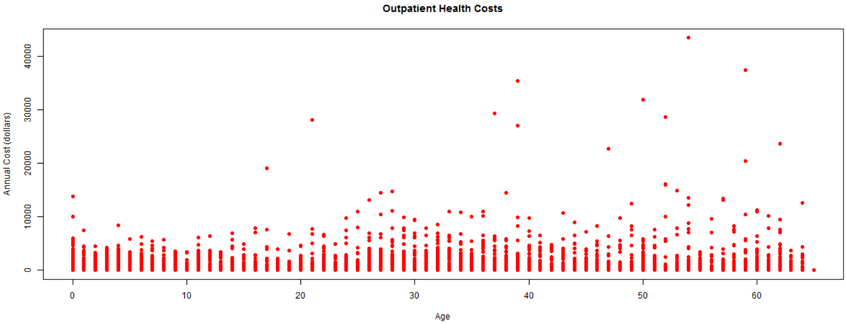
- Classical frequentist methods allow us to make exact probability statements
- Bayesian methods allow us to make uncertainty statements

Motivation
Data



Motivation
Health Care Cost Data

Data from evaluation Washington State Basic Health Plan, a program offering subsidized health insurance for low income residents (Diehr et al 1993)



Do we really need these assumptions?

- Central limit theorem guarantees asymptotically Normal sampling distribution
- Without linearity it is not even clear what quantity we are estimating
- Linearity and homoscedasticity assumptions ensure valid standard errors

Suppose we have Y and $\mathbf{X}$ sampled from a population F and

$$\mathbb{E}_F(Y|\mathbf{X}=\mathbf{x}) = \phi(\mathbf{X})$$

Define our quantity of interest as

$$\boldsymbol{\beta} = \underset{\boldsymbol{\beta}}{argmin}\ \ \mathbb{E}_F\left[\left(\phi(\mathbf{X}) - \mathbf{X}^T\beta\right)^2\right]$$

i.e. the set of coefficients minimizing average squared error in approximating the mean value of Y by linear function of $\mathbf{X}$

Definition of $\boldsymbol{\beta}$ is statement about the scientific question of interest, not details of random sampling

β is the best linear approximation to the curve $\phi(\cdot)$.

- This idea extends back to the early work of Gauss (1809), Legendre (1805), and Jacobi (1841)

Jacobi (1841) showed that

$$\beta_X = \frac{\mathrm{Cov}_F[X, Y]}{\mathrm{Var}_F[X]}$$

- Berman (1988) points out that Jacobi's result has been largely overlooked

Can also view β as solution to estimating equations (Liang and Zeger (1986))

$$\mathbb{E}_F\left[G(\beta, Y, \mathbf{X})\right] = \mathbb{E}_F\left[\mathbf{X}^T(Y - \mathbf{X}\beta)\right] = 0$$

Note that, in 1D, substituting $\mathbb{E}_F$ with summation over the sample, this gives

$$\hat{\beta}_X = \frac{Cov(X, Y)}{Var(X)}$$

Estimating equations give motivation for use of "sandwich" estimator of Huber (1967) and White (1980)

Allow for robust, frequentist inference on β in a model agnostic setting

Freedman (2006) points out the danger of fitting incorrect linear model

A Bayesian analogue of estimating equations has been an open problem for some time

Recall our definition of β

$$\boldsymbol{\beta} = \underset{\boldsymbol{\beta}}{argmin} \;\; \mathbb{E}_F \left[\left(\phi(\mathbf{X}) - \mathbf{X}^T \boldsymbol{\beta} \right)^2 \right]$$

If we knew $\phi(\cdot)$ and F, we could calculate β directly
Instead, model data using a flexible Bayesian model such that we can derive a posterior for $\phi(\cdot)$ and F
Likelihood:

$$Y|\mathbf{X} = \mathbf{x}, \phi(\cdot), \sigma^2(\cdot) \sim N(\phi(\mathbf{x}), \sigma^2(\mathbf{x}))$$

Priors: Saturated model for $\phi(\cdot)$ and $\sigma^2(\cdot)$ in discrete case, spline based priors in continuous case

The posteriors for $\phi(\cdot), \sigma^2(\cdot)$ and F *induce* a posterior for $\boldsymbol{\beta}$ from which we can use standard posterior summaries

It turns out that for random $\mathbf{X}$...

- The posterior mean for $\boldsymbol{\beta}$ is asymptotically equivalent to the ordinary least squares slope estimate $(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T Y$
- The posterior standard deviation of $\boldsymbol{\beta}$ is asymptotically equivalent to the usual sandwich estimate $\left[(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\Sigma\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\right]^{1/2}$, where Σ is a diagonal matrix of squared residuals

Conclusions

Classical assumptions of linear models not always reasonable and are not always necessary

This method provides Bayesian analogue of frequentist estimating equations and sandwich-based errors

In the random $\mathbf{X}$ case, posterior standard deviation of β is asymptotically equivalent to the "sandwich"-estimator

Payoffs for fitting very flexible Bayesian models (difficult) will be minimal when studying trends in large samples using standard sandwich-based inference (easy)

Questions?