1. (20 points). Find the solution to the initial value problem

$$\frac{dy}{dx} = \frac{cosx}{(2 + sinx)y}, y(0) = -1$$

Make sure you write your solution for y as an *explicit* function of x.

Answer: $y = -\sqrt{2\ln(2+\cancel{2}\,\sin x) + 1 - 2\ln 2}$

$$dy\, y = \frac{\cos x\, dx}{(2 + \sin x)}$$

$$\frac{y^2}{2} = \ln(2+\sin x) + c$$

$$y = \pm\sqrt{2\ln(2+\sin x) + c}$$

$$-1 = -\sqrt{2\ln 2 + c} \quad \Rightarrow c = 1 - 2\ln 2.$$

$$y = -\sqrt{2\ln(2+\sin x) + 1 - 2\ln 2}$$

2. (20 points). Solve the initial value problem
$ty' + y = cos(2t)$, $y(1) = 0$.

Make sure you write your solution for y as an *explicit* function of x.

Answer: $y = \frac{1}{t}\left(\frac{\sin 2t}{2} - \frac{\sin 2}{2}\right)$

$y' + \frac{y}{t} = + \frac{\cos 2t}{t}$ $\quad y(1)=0$

$I.F = e^{\int 1/t \, dt} = e^{\ln t} = t$

$\frac{d}{dt}(ty) = \cos 2t$

$ty = \int \cos 2t \, dt$

$ty = \frac{\sin 2t}{2} + c$; $y = \frac{1}{t}\left(\frac{\sin 2t}{2} + c\right)$

$0 = \frac{\sin 2}{2} + c$, $c = \frac{-\sin 2}{2}.$

3. (20 points). Water is pouring out of a tank at a rate proportional to the **cube root** of the amount of water present in the tank at that instant. At time $t = 0$ min there are 10 gallons of water in the tank, and at time $t = 1$ min there are 4 gallons of water in the tank. When does the tank become empty?

Answer: $10^{2/3} / 10^{2/3} - 4^{2/3} = 2.187$

$$\frac{dq}{dt} = -kQ^{1/3}$$

$$\frac{dq}{Q^{1/3}} = -k.$$

$$\frac{Q^{2/3}}{2/3} = -kt + C$$

$$\frac{10^{2/3}}{2/3} = C$$

$$\frac{4^{2/3}}{2/3} = -k + \frac{10^{2/3}}{2/3}$$

$$k = \frac{10^{2/3} - 4^{2/3}}{2/3}$$

$$-k\,t_{end} + C = 0$$

$$t_{end} = \frac{C}{k} = \frac{10^{2/3}}{10^{2/3} - 4^{2/3}}$$

4. (20 points). Consider the autonomous differential equation

$$y' = y^4 - 8y.$$

(a) Find all equilibrium solutions to the equation.

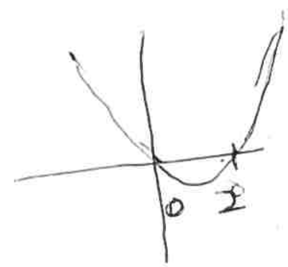
Answer: 0, 2

(b) In the space below, sketch the direction field for the equation so the all equilibrium solutions appear in your sketch. Be sure to label your axes.

(c) For each equilibrium solution in part (a) say whether it is stable, unstable, or semistable, briefly justifying each answer.

Answer: 2 is unstable, 0 is stable.

$y' = y(y-2)(y^2+2x+4)$ ← always positive.



~~Close to 2 direction field is heading away~~

~~Close to 0 direction field is heading inwards.~~

Close to 2 the direction field is heading away.

Close to 0, the direction field points inwards.

5. Consider the initial value problem

$$\frac{dy}{dt} = 2t, \; y(0) = 1.$$

(a) Use Euler's method, with step size $h = 0.05$ and starting at $t_0 = 0$, to approximate the value of the solution at $t = 0.2$.

Answer: $\boxed{1.03}$

(b) As a check, solve the equation to get the exact value of the solution at $t = 0.2$. In the box below put both the solution $y(t)$ and the exact value $y(0.2)$.

Answer: $\boxed{y = t^2+1, \; y(0.2) = 1.04}$

$$\frac{y_{n+1} - y_n}{h} = 2t_n \; ; \; y_{n+1} = y_n + h(2t_n)$$

$$y_1 = 1 + (0.05)(2(0)) = 1$$

$$y_2 = 1 + (0.05)(2(0.05))$$

$$= 1.005$$

$$y_3 = 1.005 + 0.05(2(0.1))$$

$$= 1.005 + 0.01$$

$$= 1.015$$

$$y_4 = 1.015 + 0.05(2(0.15))$$

$$= 1.015 + 0.1(0.15)$$

$$= \begin{array}{r} 1.015 \\ +\; 0.015 \\ \hline \end{array}$$

$$= 1.03$$

$$dy = 2t\,dt$$

$$y = t^2 + c \; ; \; c = 1$$