STAT 491 Midterm Exam

October 27, 2014

Instructions:

- There are five problems, for a total of 20 points.
- You are NOT allowed the use of a calculator. You may leave the answers to calculations as fractions.
- You are allowed the use of one page of handwritten notes, front and back, no larger than standard-sized paper.
- Answers must be written in the space provided in the Answer sheet. Please enter your name and Roll number at the top of the answer sheet.
  Work the problems in the sheets provided. You must enter your name and roll number here too. These must be attached.
- Turn off all cell phones and pagers.

1. (2+2 points). A fair coin is tossed four times. What is the probability that

   (a) exactly one head appears?

   (b) exactly three consecutive heads appear?

   Solution There are 2^4 elements in the sample space corresponding to length four sequences with H and T as possible entries.

   (a) The event corresponding to exactly one head is $\{HTTT, THTT, TTHT, TTTH\}$. So the probability is $4/16 = 1/4$.

   (b) The event corresponding to exactly three consecutive heads is $\{HHHT, THHH\}$. So the probability is $2/16 = 1/8$.

2. (2+2 points). Event A $\equiv$ It does not rain next Sunday.
   Event B $\equiv$ Your going out of town next Sunday. Event C $\equiv$ Your meeting a friend next Sunday. It is given that $Pr(C|AB) = 1/2, Pr(B|A) = 1/2, Pr(ABC) = 1/6$.

   (a) What is the probability that next Sunday it does not rain and you go out of town?

   (b) What is the probability that it does not rain next Sunday?

   Solution

   (a) We have, $Pr(ABC) = Pr(C|AB)Pr(AB)$. So $Pr(AB) = Pr(ABC)/Pr(C|AB) = (1/6)/(1/2) = 1/3$. We have, $Pr(ABC) = Pr(C|AB)Pr(B|A)Pr(A)$. So $Pr(A) = Pr(ABC)/Pr(C|AB)Pr(B|A) = (1/6)/(1/2 \times 1/2) = 2/3$.

3. (4 points). 10 white balls and 10 black balls are distributed in two urns in such a way that each urn contains 10 balls. At each step we draw one ball from each urn and exchange them. Let X_n be the number of white balls in the left urn at time n. Compute the transition probability for X_n, given that $X_n = 3$.

   Solution
   $X_n = 3$,
   $p(3,2) = 3/10 \times 3/10 = 9/100$;
   $p(3,3) = 3/10 \times 7/10 + 7/10 \times 3/10 = 42/100$;
   $p(3,4) = 7/10 \times 7/10 = 49/100$.
   $p(3,0) = p(3,1) = p(3,5) = \cdots = p(3,10) = 0$.

4. (4 points). A certain Markov chain has transition matrix

$$\left(\begin{array}{c|c|c} 1/2 & 1/3 & 1/6 \\ 0 & 1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{array}\right) \tag{1}$$

   If states are picked at time 0 with probability $(1/3, 1/3, 1/3)$, what is the probability distribution after one state transition?

   Solution
   The next state probability distribution is given by

$$(1/3, 1/3, 1/3)\left(\begin{array}{c|c|c} 1/2 & 1/3 & 1/6 \\ 0 & 1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{array}\right) = (5/18, 1/3, 7/18) \tag{2}$$

5. (2+2 points). The transition matrix of a certain Markov chain on states A, B, C, D, is given below.

$$\left(\begin{array}{c|c|c|c|c} & A & B & C & D \\ A & 1/2 & 1/3 & 1/6 & 0 \\ B & 0 & 1/3 & 2/3 & 0 \\ C & 1/3 & 1/3 & 0 & 1/3 \\ D & 1/3 & 1/3 & 0 & 1/3 \end{array}\right) \tag{3}$$

   Compute

   (a) $p^2(A, B)$
   (b) $p^2(B, D)$.

   Solution

   (a)
$$p^2(A,B) = p(A,A) \times p(A,B) + p(A,B) \times p(B,B) + p(A,C) \times p(C,B) = 1/6 + 1/9 + 1/18 = 1/3.$$

   (b)
$$p^2(B,D) = p(B,C) \times p(C,D) = 2/9.$$