STAT 491 Midterm Exam

October 27, 2014

Instructions:

- There are five problems, for a total of 20 points.
- You are NOT allowed the use of a calculator. You may leave the answers to calculations as fractions.
- You are allowed the use of one page of handwritten notes, front and back, no larger than standard-sized paper.
- Please enter your name and Roll number at the top of the answer sheet.
  Answers and only answers must be written in the space provided in the Answer sheet. Please do not display working on the answer sheet.
- Work the problems in the sheets provided. You must enter your name and roll number here too. These must be attached. Answers, without some evidence in these sheets of how they were arrived at, will get reduced credits.
- Turn off all cell phones and pagers.

1. (7 points) The matrix P specified below is the transition matrix of a Markov chain.

$$P = \begin{pmatrix} & 1 & 2 & 3 & 4 & 5 \\ 1 & 2/3 & 1/3 & 0 & 0 & 0 \\ 2 & 2/3 & 0 & 1/3 & 0 & 0 \\ 3 & 0 & 2/3 & 0 & 1/3 & 0 \\ 4 & 0 & 0 & 2/3 & 0 & 1/3 \\ 5 & 0 & 0 & 0 & 2/3 & 1/3 \end{pmatrix} \tag{1}$$

(a) Which states are transient and which, recurrent?

(b) What is the period of state 2?

(c) What is the probability of returning to state 2 if we start from 2?

(d) What is the probability of reaching state 1 from state 2?

(e) What is its stationary distribution?

(f) Let N_2 denote the number of times the chain returns to 2 in time n. What is $\lim_{n\to\infty} E(N_2)/n$?

Solution

(a) All states are recurrent since there are directed paths between any ordered pair of vertices.

(b) State 1 has period 1 because of the selfloop present on it. Since all states can reach and be reached from state 1, all states have period 1. So state 2 has period 1.

(c) This is an irreducible Markov chain. So the probability of returning to a state, starting from it is 1.

(d) For the same reason as above this probability is also 1.

(e) There are no directed loops except those made up of parallel but oppositely directed edges. Therefore this Markov chain is reversible. We have

$$\pi_i \times 1/3 = \pi_{i+1} \times 2/3, i = 1, 2, 3, 4.$$

So the π_i are proportional to $(16, 8, 4, 2, 1)$. Hence $\pi = (16/31, 8/31, 4/31r, /2/31, 1/31)$.

(f) $\lim_{n\to\infty} E(N_2)/n = \pi_2 = 8/31$.

2. (2 points) The matrix P specified below is the transition matrix of a Markov chain. What is its stationary distribution?

$$P = \begin{pmatrix} & 1 & 2 & 3 & 4 \\ 1 & 1/2 & 1/4 & 1/8 & 1/8 \\ 2 & 1/4 & 1/2 & 1/8 & 1/8 \\ 3 & 1/8 & 1/8 & 0 & 3/4 \\ 4 & 1/8 & 1/8 & 3/4 & 0 \end{pmatrix} \tag{2}$$

Solution

This is a doubly stochastic matrix (rows and columns add up to 1.) So adding all rows will give a row of 1s. So the unique stationary distribution $\pi = (1/4, 1/4, 1/4, 1/4)$.

3. (5 points) The matrix P specified below is the transition matrix of a Markov chain.

$$P = \begin{pmatrix} & A & B & C \\ A & 1/4 & 2/3 & 1/12 \\ B & 1/4 & 2/3 & 1/12 \\ C & 0 & 0 & 1 \end{pmatrix} \quad (3)$$

(a) Write equations for the probabilities $q(i, C)$ of reaching from states A, B to state C.

(b) Write equations for the expected time $l(i, C)$ of reaching from states A, B to state C.

(c) What is $q(A, C)$?

(d) What is $l(B, C)$?

Solution

(a)

$$q(A, C) = 1/12 + 1/4q(A, C) + 2/3q(B, C)$$
$$q(B, C) = 1/12 + 1/4q(A, C) + 2/3q(B, C)$$

Equivalently,

$$3/4q(A, C) - 2/3q(B, C) = 1/12$$
$$-1/4q(A, C) + 1/3q(B, C) = 1/12$$

(b)

$$l(A, C) = 1 + 1/4l(A, C) + 2/3l(B, C)$$
$$l(B, C) = 1 + 1/4l(A, C) + 2/3l(B, C)$$

Equivalently,

$$3/4l(A, C) - 2/3l(B, C) = 1$$
$$-1/4l(A, C) + 1/3l(B, C) = 1.$$

(c) $q(A, C) = 1$.

(d) $l(B, C) = 12$.

4. (2 points) Write the GCD of $94, 100$ in the form $94a + 100b$.

Solution

$$16 \times 100 + (-17) \times 94.$$

5. (4 points) A certain branching process has the following one step probability for number of progeny:
$p_0 = 1/6, p_1 = 1/6, p_2 = 2/3$.

(a) Write down the equation the probability of extinction satisfies.

(b) What is the probability of extinction?

Solution

(a)

$$\rho = p_0 + p_1\rho^1, \cdots p_k\rho^k \cdots = 1/6 + 1/6\rho + 2/3\rho^2.$$

Equivalently,

$$1/6 - 5/6\rho + 2/3\rho^2 = 0,$$

or

$$1 - 5\rho + 4\rho^2 = 0,$$

(b) This is the smallest nonnegative root of the above equation. The roots are $1/4, 1$. So the probability of extinction is $1/4$.

Answer Sheet

Name Roll Number

- Q1 (a) Q1(b)
- Q1 (c) Q1(d)
- Q1 (e)
- Q1 (f)
- Q2
- Q3(a)
- Q3(b)
- Q3(c) Q3(d)
- Q4
- Q5 (a) Q5(b)