Name: KEY

Math 307D Final Exam

March 20, 2013

Instructions: There are ten problems, with the value of each problem indicated, for a total of 160 points. You are allowed the use of one page of handwritten notes, front and back, on standard size paper. You are also allowed use of a scientific calculator (but graphing calculators and other calculational devices are not allowed). The next page of the exam contains a table of Laplace transforms, which you may use in your solutions.

- Work the problems in the space provided. If you need more space, use the back of the page, and clearly indicate that you are doing so.
- Neatness counts! A well-organized solution, even with mistakes, will get more partial credit than a haphazard collection of unrelated calculations.
- Put the answer you want considered in the **Box** provided.
- You **MUST** show all your work and reasoning to receive credit. If in doubt, ask for clarification.
- Turn off all cell phones and pagers.

Problem 1	15 points	
Problem 2	15 points	
Problem 3	15 points	
Problem 4	15 points	
Problem 5	15 points	
Problem 6	20 points	
Problem 7	15 points	
Problem 8	15 points	
Problem 9	15 points	
Problem 10	20 points	
Total	160 points	

1. (15 points). Solve the initial value problem

$$y' = \frac{x+1}{x^2(2y+1)}, \quad y(1) = 0.$$

Your solution should give y *explicitly* as a function of x.

Answer: $\boxed{y = \frac{1}{2}\left(-1 + \sqrt{5 - \frac{4}{x} + 4\ln x}\right)}$

Separable: $\frac{dy}{dx} = \frac{x+1}{x^2}\,\frac{1}{2y+1} \Rightarrow (2y+1)\,dy = \left(\frac{x+1}{x^2}\right)dx$

$\int$: $\int (2y+1)\,dy = \int \left(\frac{1}{x} + \frac{1}{x^2}\right)dx$

$$y^2 + y = \ln x - \frac{1}{x} + C_1$$

$y(1)=0$: $0^2 + 0 = 0 - 1 + C_1 \Rightarrow C_1 = 1$

$$y^2 + y = \ln x - \frac{1}{x} + 1$$

Solve for y using quadratic formula:

$$y^2 + y + \left(\frac{1}{x} - \ln x - 1\right) = 0$$

$$y = \frac{-1 \pm \sqrt{1 - 4\left(\frac{1}{x} - \ln x - 1\right)}}{2}$$

$$= \frac{-1 \pm \sqrt{5 - \frac{4}{x} + 4\ln x}}{2}$$

Use $y(1)=0$ to pick correct sign:

$0 = y(1) = \frac{-1 \pm \sqrt{5-4}}{2} = \frac{-1 \pm 1}{2}$, so $\underset{=}{+}$

$$y = \frac{1}{2}\left(-1 + \sqrt{\cancel{4}5 - \frac{4}{x} + 4\ln x}\right)$$

2. (15 points) Solve the initial value problem

$$(t^2+1)y' + (2t)y = t\,e^t, \quad y(0) = 2.$$

Answer: $\boxed{y = \dfrac{te^t - e^t + 3}{t^2+1}}$

Standard form: $y' + \left(\frac{2t}{t^2+1}\right)y = \frac{t}{t^2+1}e^t$

Integrating factor $\mu(t) = e^{\int \frac{2t}{t^2+1}dt}$

Use substitution $u = t^2+1$, $du = 2t\,dt$

$$\int \frac{2t}{t^2+1}dt = \int \frac{du}{u} = \ln u = \ln(t^2+1)$$

$$\mu(t) = e^{\ln(t^2+1)} = t^2+1$$

So

$$[(t^2+1)y]' = te^t$$

$$(t^2+1)y = \int te^t dt \overset{\text{parts}}{=} te^t - \int e^t dt$$
$$= te^t - e^t + C_1$$

$y(0) = 2 \Rightarrow$

$$2 = -1 + C_1 \Rightarrow C_1 = 3$$

$$(t^2+1)y = te^t - e^t + 3$$

$$y = \frac{te^t - e^t + 3}{t^2+1}$$

3. (15 points) A population of bacteria increases at a rate proportional to the square root of the current population. At time $t = 0$ days the population is 100, and at time $t = 4$ the population is 900.

(a) Find a formula for the population $P(t)$ at time t days.

(b) At what time does the population reach 3600?

Answer: (a) $P(t) = (5t+10)^2$, (b) 10 days

(a) Let $P(t)$ = population at time t days.

$$\frac{dP}{dt} \propto \sqrt{P}, \text{ or } \frac{dP}{dt} = k\sqrt{P} \qquad P(0) = 100, \ P(4) = 900$$

Separate varaables: $\frac{dP}{\sqrt{P}} = k\,dt$

Integrate: $\int \frac{dP}{\sqrt{P}} = \int k\,dt \Rightarrow 2\sqrt{P} = kt + C$

$P(0) = 100 \Rightarrow \quad 20 = C$

$P(4) = 900 \Rightarrow \quad 2\cdot 30 = 4k + 20 \Rightarrow 40 = 4k \Rightarrow k = 10$

$$2\sqrt{P(t)} = 10t + 20$$

$$\Rightarrow \sqrt{P(t)} = 5t + 16$$

$$\Rightarrow P(t) = (5t+10)^2$$

(b) $P(t_0) = 3600 \Rightarrow (5t_0 + 10)^2 = 3600$

$\Rightarrow 5t_0 + 10 = 60$

$\Rightarrow t_0 = 10$ days

4. (15 points) Solve the initial value problem

$$y'' + 2y' + 5y = 0, \quad y(0) = 2,\ y'(0) = 4.$$

Answer: $\boxed{y(t) = 2e^{-t}\cos(2t) + 3e^{-t}\sin(2t)}$

Char eqn: $r^2 + 2r + 5 = 0$

Roots: $\dfrac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i, \quad \lambda = -1, \mu = 2$

General solution:

$$y = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$$

$$2 = y(0) = c_1 \Rightarrow c_1 = 2$$

$$y'(t) = 2\left[e^{-t}(-2\sin(2t)) - e^{-t}\cos(2t)\right] + c_2\left[e^{-t}(2\cos(2t)) - e^{-t}\sin(2t)\right]$$

$$4 = y'(0) = -2 + 2c_2 \Rightarrow c_2 = 3$$

$$\therefore \quad y(t) = 2e^{-t}\cos(2t) + 3e^{-t}\sin(2t)$$

5. (15 points) Find the general solution to

$$y'' + y' - 2y = e^t + \sin t$$

Answer: $\boxed{y = c_1 e^t + c_2 e^{-2t} + \frac{1}{3} t e^t - \frac{1}{10}\cos t - \frac{3}{10}\sin t}$

Char equ: $r^2 + r - 2 = 0 \Rightarrow (r+2)(r-1) = 0 \Rightarrow r = -2$ or $r = 1$

$\therefore\ y_c(t) = c_1 e^t + c_2 e^{-2t}$

$g_1(t) = e^t$, particular solution has form $y_{p_1} = A t e^t$

since e^t satisfies homogeneous equ.

$$y_{p_1}' = A(te^t + e^t), \quad y_{p_1}'' = A(te^t + 2e^t)$$

$$y_{p_1}'' + y_{p_1}' - 2y_{p_1} = A(\cancel{te^t} + 2e^t) + A(\cancel{te^t} + e^t) - 2\cancel{A}(te^t)$$

$$= 3Ae^t \stackrel{?}{=} e^t \Rightarrow A = \frac{1}{3}$$

$g_2(t) = \sin t$ has $y_{p_2} = B\cos t + C\sin t$,

$$y_{p_2}'(t) = -B\sin t + C\cos t, \quad y_{p_2}'' = -B\cos t - C\sin t$$

$$y_{p_2}'' + y_{p_2}' - 2y_{p_2} = (-B\cos t - C\sin t) + (-B\sin t + C\cos t)$$

$$- 2(B\cos t + C\sin t) \stackrel{?}{=} \sin t$$

$$\Rightarrow \underbrace{[-B + C - 2B]}_{=0}(\cos t) + \underbrace{[-C - B - 2C]}_{=1}\sin t = \sin t$$

$$\Rightarrow \begin{cases} -3B + C = 0 \\ -B - 3C = 1 \end{cases} \Rightarrow B = -\frac{1}{10},\ C = -\frac{3}{10}$$

$$y_{p_2} = -\frac{1}{10}\cos t - \frac{3}{10}\sin t$$

6. (20 points) A 10 lb weight stretches a spring 2 ft. Suppose the weight is pulled down an additional foot and given a downward velocity of 2 ft/sec. There is no damping, nor are there external forces. Determine the amplitude of the subsequent motion.

Answer: $R = \sqrt{\frac{5}{4}}$

$mg = 10$ lb $\Rightarrow$ $m(32) = 10 \Rightarrow m = \frac{10}{32} = \frac{5}{16}$

$mg = kL \Rightarrow 10 = k \cdot 2 \Rightarrow k = 5$

So the equation of motion is

$$\frac{5}{16} u'' + 5u = 0, \quad u(0) = 1, \; u'(0) = 2$$

$$\Rightarrow u'' + 16u = 0$$

Char equ: $r^2 + 16 = 0 \Rightarrow r = \pm 4$, so gen soln is

$$u(t) = c_1 \cos(4t) + c_2 \sin(4t)$$

$$1 = u(0) = c_1$$

$$u'(t) = -4\sin(4t) + 4c_2 \cos(4t)$$

$$2 = u'(0) = 4c_2 \Rightarrow c_2 = \frac{1}{2}$$

$\therefore$ $u(t) = \cos(4t) + \frac{1}{2}\sin(4t)$

Amplitude $R = \sqrt{1^2 + \left(\frac{1}{2}\right)^2} = \sqrt{5/4}$

7. (15 points) Let $f(t)$ be a function whose Laplace transform is $F(s)$. Define a new function

$$g(t) = e^{-2t} f(3t).$$

Determine the Laplace transform $G(s)$ of $g(t)$ in terms of $F(s)$.

Answer: $\boxed{G(s) = \frac{1}{3} F\left(\frac{s+2}{3}\right)}$

Method I (use tables): Let $h(t) = f(3t)$. By (#15) in table,

$$H(s) = \frac{1}{3} F\left(\frac{s}{3}\right).$$

$g(t) = e^{-2t} h(t)$, so by (#14) in table

$$G(s) = H(s+2) = \frac{1}{3} F\left(\frac{s+2}{3}\right)$$

Method II (use definition of Laplace transform):

$$G(s) = \int_0^\infty e^{-st} g(t)\,dt = \int_0^\infty e^{-st} e^{-2t} f(3t)\,dt$$

$$= \int_0^\infty e^{-(s+2)t} f(3t)\,dt \qquad \text{use } u = 3t, \; du = 3dt$$

$$= \int_0^\infty e^{-(s+2)\left(\frac{u}{3}\right)} f(u) \left(\tfrac{1}{3} du\right)$$

$$= \frac{1}{3} \int_0^\infty e^{-\left(\frac{s+2}{3}\right) u} f(u)\,du$$

$$= \frac{1}{3} F\left(\frac{s+2}{3}\right)$$

8. (15 points) Find the inverse Laplace transform of

$$F(s) = \frac{2s-3}{s^2+2s+10}.$$

Answer: $\boxed{f(t) = 2e^{-t}\cos(3t) - \frac{5}{3}e^{-t}\sin(3t)}$

$$s^2+2s+10 = (s+1)^2+3^2$$

$$\frac{2s-3}{s^2+2s+10} = \frac{2(s+1)-5}{(s+1)^2+3^2}$$

$$\frac{s+1}{(s+1)^2+3^2} \xrightarrow{\mathcal{L}^{-1}} e^{-t}\cos(3t) \quad \text{(Table \#10)}$$

$$\frac{3}{(s+1)^2+3^2} \xrightarrow{\mathcal{L}^{-1}} e^{-t}\sin(3t) \quad \text{(Table \#9)}$$

$$\frac{2s-3}{s^2+2s+10} \longrightarrow 2\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+3^2}\right) - \frac{5}{3}\mathcal{L}^{-1}\left(\frac{3}{(s+1)^2+3^2}\right)$$

$$= 2e^{-t}\cos(3t) - \frac{5}{3}e^{-t}\sin(3t)$$

9. (15 points). Use Laplace transforms to solve the initial value problem

$$y'' - y' - 6y = 0, \quad y(0) = 1,\ y'(0) = -1.$$

You can *check your answer*!

Answer: $y(t) = \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}$

Take $\mathcal{L}$:

$$s^2 Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) - 6Y(s) = 0$$

$$\Rightarrow \quad (s^2 - s - 6)Y(s) = s - 1 - 1 = s - 2$$

$$Y(s) = \frac{s-2}{s^2-s-6} = \frac{s-2}{(s-3)(s+2)}$$

$$= \frac{A}{s-3} + \frac{B}{s+2} = \frac{1/5}{s-3} + \frac{4/5}{s+2} \quad \text{(cover-up!)}$$

$$y(t) = \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}$$

Check: e^{3t}, e^{-2t} satisfy equ, so $y(t)$ does.

$$y(0) = \frac{1}{5} + \frac{4}{5} \ \checkmark$$

$$y'(t) = \frac{3}{5}e^{3t} - \frac{8}{5}e^{-2t}$$

$$y'(0) = \frac{3}{5} - \frac{8}{5} = -1 \ \checkmark$$

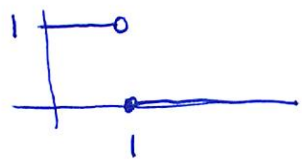
10. (20 points) Let $f(t)$ be the forcing function defined by $f(t) = 1$ if $0 \leq t \leq 1$, and $f(t) = 0$ if $t > 1$. Solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 0, \; y'(0) = 0.$$

Answer:

$$\boxed{\begin{array}{l} y(t) = h(t) - u_1(t)\, h(t-1) \\ \text{where } h(t) = 1 - \cos t \end{array}}$$

$f(t)$:



$\Rightarrow f(t) = 1 - u_1(t)$

Take $\mathcal{L}$:

$$s^2 Y(s) + Y(s) = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$Y(s) = \frac{1}{s(s^2+1)} - e^{-s} \frac{1}{s(s^2+1)}$$

Let $H(s) = \dfrac{1}{s(s^2+1)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+1}$

$$\Rightarrow \quad 1 = A(s^2+1) + (Bs+C)s$$

$$1 = (A+B)s^2 + Cs + A$$

$$\Rightarrow A = 1, \; B = -1, \; C = 0$$

$$H(s) = \frac{1}{s} - \frac{s}{s^2+1} \xrightarrow{\mathcal{L}^{-1}} 1 - \cos t = h(t)$$

$$e^{-s} H(s) \xrightarrow{\mathcal{L}^{-1}} u_1(t)\, h(t-1)$$

$$y(t) = h(t) - u_1(t)\, h(t-1)$$