Stat-491-Fall2014-Assignment-II

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1 Note

These problems are from Durrett's book.
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1. A fair coin is tossed repeatedly with results $Y_0, Y_1, Y_2, \cdots$, that are 0 or 1 with probability 1/2 each. For $n \geq 1$, let $X_n = Y_n + Y_{n-1}$ be the number of $1s$ in the $(n-1)^{th}$ and n^{th} tosses. Is X_n a Markov chain?

   Solution
   We need $Pr\{X_{n+1} = j \mid X_n = i, X_{n-1} = i'\} = Pr\{X_{n+1} = j \mid X_n = i\},\ for\ all\ i, i', j$, for X_n to be a Markov Chain.
   Let $j = 2, i = 1, i' = 0$. We have $Pr\{X_{n+1} = j \mid X_n = i, X_{n-1} = i'\} = 1/2$, since

   - the only way we can have $j = 2, i = 1, i' = 0$ is to have $Y_{n+1} = 1, Y_n = 1, Y_{n-1} = 0, Y_{n-2} = 0$,
   - $i = 1, i' = 0$ fix $Y_n = 1, Y_{n-1} = 0, Y_{n-2} = 0$, and
   - Y_{n+1} is 1 or 0 with probability 1/2.

   Now let us keep $j = 2, i = 1$, but change i' to 2. In this case we must have $Y_n = 0, Y_{n-1} = 1, Y_{n-2} = 1$. But this means $X_{n+1} \neq 2$. So $Pr\{X_{n+1} = j \mid X_n = i, X_{n-1} = i'\} = 0$.
   Thus $Pr\{X_{n+1} = j \mid X_n = i, X_{n-1} = i'\} \neq Pr\{X_{n+1} = j \mid X_n = i\},\ for\ all\ i, i', j$. We conclude X_n is not a Markov chain.

2. Five white balls and five black balls are distributed in two urns in such a way that each urn contains five balls. At each step we draw one ball from each urn and exchange them. Let X_n be the number of white balls in the left urn at time n. Compute the transition probability for X_n.

   Solution
   There are 6 states corresponding to $0, 1, 2, 3, 4, 5$, white balls in the left urn. The transition probabilities are as follows:

   $$p(5,4) = 1, p(4,4) = 8/25, p(4,5) = 1/25, p(4,3) = 16/25, p(3,3) = 12/25, p(3,4) = 4/25, p(3,2) = 9/25,$$

   $$p(2,2) = 12/25, p(2,3) = 9/25, p(2,1) = 4/25, p(1,1) = 8/25, p(1,2) = 16/25, p(1,0) = 1/25, p(0,1) = 1.$$

   We solve a token case, $p(3,4)$. In this case we need to move from 3 white balls in the left urn to 4 white balls. This can be done by picking a black ball in the left urn (with probability 2/5) and exchanging it with a white ball in the right urn (picking it with probability 2/5). So the probability of moving from state 3 to state 4 is $2/5 \times 2/5 = 4/25$.

   Note that the outgoing probabilities from any state add up to 1. Further there is a symmetry between the sequences $5, 4, 3, 2, 1, 0$ and $0, 1, 2, 3, 4, 5$, corresponding to the symmetry between 'white' and 'black'.

3. We repeatedly roll two four sided dice with numbers $1, 2, 3$, and 4 on them. Let Y_k be the sum on the k^{th} roll, $S_n = Y_1 + \cdots + Y_n$ be the total of the first n rolls, and let $X_n = S_n\ (mod\ 6)$. Find the transition probability for X_n.

Solution
The states are $0, 1, 2, 3, 4, 5$ *mod* 6. The numbers on the two dice together can give the following sample space:

$$(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4).$$

Each of the elements of the sample space has a probability $1/16$. Translated into $mod-6$ arithmetic, the sample space element pair sums become, respectively,

$$2,3,4,5,3,4,5,0,4,5,0,1,5,0,1,2.$$

We therefore have the probabilities for the $mod-6$ elements as follows:

$$p(0) = 3/16, p(1) = 2/16, p(2) = 2/16, p(3) = 2/16, p(4) = 3/16, p(5) = 4/16.$$

If $X_n = x\ mod - 6$, then $Pr\{X_{n+1} = x + y\ mod - 6 \mid X_n = x\ mod - 6\} = p(y)$. So $p(x, x) = 3/16, p(x, x+1) = 2/16, p(x, x+2) = 2/16, p(x, x+3) = 2/16, p(x, x+4) = 3/16, p(x, x+5) = 4/16$.

4. The 1990 census showed that 36% of the households in the District of Columbia were homeowners while the remainder were renters. During the next decade 6% of the homeowners became renters and 12% of the renters became homeowners. What percentage were homeowners in 2000? in 2010?

   Solution
   Let H, R denote homeowners and renters respectively.

   $$p_{old}(H) = 0.36, p_{old}(R) = 0.64, p(R \mid H) = 0.06, p(H \mid H) = 0.94, p(R \mid R) = 0.88, p(H \mid R) = 0.12.$$

   We therefore have
   $$\begin{aligned} p_{new}(R) &= p(R \mid H) \times p_{old}(H) + p(R \mid R) \times p_{old}(R) \\ &= .06 \times 0.36 + 0.88 \times 0.64 = 0.5848; \\ p_{new}(H) &= p(H \mid R) \times p_{old}(R) + p(H \mid H) \times p_{old}(H) \\ &= 0.12 \times 0.64 + 0.94 \times 0.36 = 0.4152. \end{aligned}$$

   For the decade $2000 - 2010$

   $$p_{old}(H) = 0.4152, p_{old}(R) = 0.5848, p(R \mid H) = 0.06, p(H \mid H) = 0.94, p(R \mid R) = 0.88, p(H \mid R) = 0.12.$$

   We therefore have
   $$\begin{aligned} p_{new}(R) &= p(R \mid H) \times p_{old}(H) + p(R \mid R) \times p_{old}(R) \\ &= .06 \times 0.4152 + 0.88 \times 0.5848 = 0.539536; \\ p_{new}(H) &= p(H \mid R) \times p_{old}(R) + p(H \mid H) \times p_{old}(H) \\ &= 0.12 \times 0.5848 + 0.94 \times 0.4152 = 0.460462. \end{aligned}$$

   We check that $p(H) + p(R) = 1$, in each case.

5. Consider a gamblers ruin chain with $N = 4$. That is, if $1 \le i \le 3, p(i, i+1) = 0.4$, and $p(i, i-1) = 0.6$, but the endpoints are absorbing states: $p(0, 0) = 1$ and $p(4, 4) = 1$. Compute $p^3(1, 4)$ and $p^3(1, 0)$.

   Solution
   The only way of going from 1 to 4 in 3 steps is the path $1 \to 2, 2 \to 3, 3 \to 4$.
   So $p^3(1, 4) = 0.4 \times 0.4 \times 0.4 = 0.064$.
   The only ways of going from 1 to 4 in 3 steps are the paths $1 \to 0, 0 \to 0, 0 \to 0$, and
   $1 \to 2, 2 \to 1, 1 \to 0$.
   The corresponding probabilities are $0.6 \times 1 \times 1 = 0.6$ and $0.4 \times 0.6 \times 0.6 = 0.144$, which add up to 0.744.

6. A taxicab driver moves between the airport A and two hotels B and C according to the following rules. If he is at the airport, he will be at one of the two hotels next with equal probability. If at a hotel then he returns to the airport with probability 3/4 and goes to the other hotel with probability 1/4.
(a) Find the transition matrix for the chain.
(b) Suppose the driver begins at the airport at time 0. Find the probability for each of his three possible locations at time 2 and the probability he is at hotel B at time 3.

Solution
$p(A, A) = 0, p(A, B) = 0.5, p(A, C) = 0.5;$
$p(B, A) = 3/4, p(B, B) = 0, p(B, C) = 1/4;$
$p(C, A) = 3/4, p(C, B) = 1/4, p(C, C) = 0;$
$p^2(A, A) = 6/8, p^2(A, B) = 1/8, p^2(A, C) = 1/8, p^3(A, B) = 13/32.$

7. Suppose that the probability it rains today is 0.3 if neither of the last two days was rainy, but 0.6 if at least one of the last two days was rainy. Let the weather on day n, W_n, be R for rain, or S for sun. W_n is not a Markov chain, but the weather for the last two days $X_n = (W_{n-1}, W_n)$ is a Markov chain with four states RR, RS, SR, SS.
(a) Compute its transition probability.
(b) Compute the two-step transition probability.
(c) What is the probability it will rain on Wednesday given that it did not rain on Sunday or Monday.

Solution
The transition matrix is

	RR	RS	SR	SS
RR	0.6	0.4	0.0	0.0
RS	0.0	0.0	0.6	0.4
SR	0.6	0.4	0.0	0.0
SS	0.0	0.0	0.3	0.7

The two step transition matrix is

	RR	RS	SR	SS
RR	.36	.24	.24	.16
RS	.36	.24	.12	.28
SR	.36	.24	.24	.16
SS	.18	.12	.21	.49

We are given that starting state is SS. and asked to compute $p(R)$ after two steps. This could happen as RR or SR corresponding to Tuesday and Wednesday

$$Pr\{X_2 = SR \mid X_0 = SS\} + Pr\{X_2 = RR \mid X_0 = SS\} = 0.21 + 0.18 = 0.39.$$