Random Sums

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December 8, 2014

Theorem:
Let $Y_1, Y_2 \cdots$ be independent and identically distributed,let N be an independent non-negative integer valued random variable, and let $S = Y_1 + \cdots + Y_N$ with $S = 0$ when $N = 0$.

1. If $E[|Y_i|], E[N] < \infty, then\ E[S] = E[N].E[Y_i]$.
2. If $E[Y_i^2], E[N^2] < \infty, then\ var(S) = E[N]var(Y_i) + var(N)(E[Y_i])^2$.
3. If N is $Poisson(\lambda)$, then $var(S) = \lambda E[Y_i^2]$.

Proof:

1. We have
$$E[S] = \Sigma_{n=0}^{\infty} E[S \mid N = n] \times P(N = n)$$
$$= \Sigma_{n=0}^{\infty} nE[Y_i] \times P(N = n) = E[N]E[Y_i].$$
2. $var[S] \equiv E[S^2] - (E[S])^2$. If $= Y_1 + \cdots + Y_n, Y_i$ i.i.d., then $var[S] = n(var[Y_i])$. Now,
$E[S^2 \mid N = n] = var[S \mid N = n] + (E[S \mid N = n])^2 = n\ var[Y_i] + (nE[Y_i])^2$. Therefore
$$E[S^2] = \Sigma_{n=0}^{\infty} E[S^2 \mid N = n] \times P(N = n)$$
$$= \Sigma_{n=0}^{\infty} (n\ var[Y_i] + n^2(E[Y_i])^2) \times P(N = n)$$
$$= (E[N])var[Y_i] + (E[Y_i])^2 \times E[N^2].$$
Therefore
$$var[S] \equiv E[S^2] - (E[S])^2 = (E[N])var[Y_i] + (E[Y_i])^2 \times E[N^2] - (E[N]E[Y_i])^2$$
$$= (E[N])var[Y_i] + (E[Y_i])^2 \times (E[N^2] - (E[N])^2)$$
$$= (E[N])var[Y_i] + var[N](E[Y_i])^2.$$
3. If N is $Pois(\lambda)$, then $E[N] = var[N] = \lambda$. So
$$var[S] = \lambda(var[Y_i] + (E[Y_i])^2) = \lambda(E[Y_i^2]).$$