Name: KEY

Math 307C Second Midterm Exam

November 22, 2010

Instructions: There are five problems, for a total of 100 points. You are allowed the use of a scientific calculator (but graphing calculators and other calculational devices are **NOT ALLOWED**). You are also allowed the use of one page of hand-written notes (front and back), no larger than standard-sized paper.

- Work the problems in the space provided. If you need more space, use the back of the page, and clearly indicate that you are doing so.
- Neatness counts! A well-organized solution, even with mistakes, will get more partial credit than a haphazard collection of unrelated calculations.
- Put the answer you want considered in the $\boxed{\textbf{Box}}$ provided.
- You **MUST** show all your work and reasoning to receive credit. If in doubt, ask for clarification.
- Turn off all cell phones and pagers.

Problem 1	20 points	
Problem 2	20 points	
Problem 3	20 points	
Problem 4	20 points	
Problem 5	20 points	
Total	100 points	

1. (20 points). Find the solution to the initial value problem

$$y'' + 2y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = 4$$

Answer: $y(t) = 2e^{-t}\cos t + 6e^{-t}\sin t$

Char eqn: $r^2 + 2r + 2 = 0$

Roots: $\frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i, \quad \lambda = -1, \ \mu = 1$

General soln is

$$y = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$$

$$2 = y(0) = c_1 \Rightarrow \boxed{c_1 = 2}$$

$$y'(t) = 2\left[e^{-t}(-\sin t) - e^{-t}\cos t\right] + c_2\left[e^{-t}\cos t - e^{-t}\sin t\right]$$

$$4 = y'(0) = -2 + c_2 \Rightarrow \boxed{c_2 = 6}$$

So $y(t) = 2e^{-t}\cos t + 6e^{-t}\sin t$

2. (20 points) Find the general solution to

$$y'' + 4y' + 4y = e^{-2t}$$

Answer: $\boxed{y(t) = c_1 e^{-2t} + c_2 t e^{-2t} + \frac{1}{2} t^2 e^{-2t}}$

Char eqn: $r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0$

So $r = -2$ is a real, repeated root.

So the general solution to the corresponding homogeneous eqn is

$$y_c(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

Since both e^{-2t} and te^{-2t} are solns to homog eqn, the correct form for using undet'd coeff's is

$$y_p = A t^2 e^{-2t}$$

$$y_p' = A t^2(-2e^{-2t}) + 2tAe^{-2t}$$

$$y_p'' = (-2A)\left[t^2(-2e^{-2t}) + 2te^{-2t}\right] + (2A)\left[t(-2e^{-2t}) + e^{-2t}\right]$$

$$y_p'' + 4y_p' + 4y_p = \overbrace{(4A - 8A + 4A)}^{0} t^2 e^{-2t} + \overbrace{(-8A + 8A)}^{0} t e^{-2t} + (2A)e^{-2t} = e^{-2t} \Rightarrow A = \frac{1}{2}$$

3. (20 points). Let $y(t)$ be the solution to the initial value problem

$$y'' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 2.$$

Define a new function $E(t) = 4y(t)^2 + [y'(t)]^2$. Show that $E(t)$ is a constant function, that is, it does not vary with t.

[Here $E(t)$ represents the energy of the system, and you are showing that this system obeys the law of conservation of energy.]

$\boxed{\text{First solution}}$ $\quad E'(t) = \frac{d}{dt}\left[4y(t)^2 + [y'(t)]^2\right]$

$$= 8y(t)y'(t) + 2y'(t)y''(t)$$

But $y''(t) = -4y(t)$ since $y(t)$ is a solution, so

$$E'(t) = 8y(t)y'(t) + 2y'(t)[-4y(t)]$$
$$= (8-8)\,y(t)y'(t) \equiv 0$$

Since $E'(t) \equiv 0$, $E(t)$ must be constant.

$\boxed{\text{Second solution}}$ Can solve the IVP directly

to get $y(t) = \cos(2t) + \sin(2t)$. Then

$y'(t) = -2\sin(2t) + 2\cos(2t)$, so

$$E(t) = 4y(t)^2 - [y'(t)]^2 = 4\left[\cos^2(2t) + \cancel{2\cos(2t)\sin(2t)} + \sin^2(2t)\right)$$
$$+ 4\sin^2(2t) - \cancel{8\sin(2t)\cos(2t)} + 4\cos^2(2t)$$
$$= 4(\cos^2(2t) + \sin^2(2t)) + 4(\sin^2(2t) + \cos^2(2t))$$
$$= 4 + 4 = 8$$

(This solution tells you what the actual constant is)

4. (20 points). The equation

$$t^2y'' - 4t\,y' + 6y = 0$$

has $y_1(t) = t^2$ as one solution. Find a second linearly independent solution.

Answer: $\boxed{y_2(t) = t^3}$

$y_2(t) = t^2 v(t) = t^2 v$

$y_2'(t) = t^2 v'(t) + 2t v(t)$

$y_2''(t) = t^2 v'' + 4t v' + 2v$

$$t^2 y_2'' - 4t y_2' + 6y_2 = t^2(t^2 v'' + 4t v' + 2v)$$
$$- 4t(t^2 v' + 2t v)$$
$$+ 6t^2 v$$
$$= t^4 v'' + \overset{0}{(\cancel{4t^3 - 4t^3})} v' + \overset{0}{(\cancel{2t^2 - 8t^2 + 6t^2})} = 0$$

$\therefore$ $\cancel{t^4} v'' = 0$

$\Rightarrow v'' = 0$

$\Rightarrow v' = c$

$\Rightarrow v = ct + d$

To get a second solution, use $c=1$, $d=0$, so $v(t) = t$ and

$$y_2(t) = t^2(t) = t^3$$

5. (20 points). A mass weighing 32 lb stretches a spring 2 ft. There is no damping. The mass is initially at rest. At time $t = 0$ it is suddenly set in motion by an external force of $F(t) = \cos t$. Determine the displacement function $u(t)$ for the subsequent motion. In this problem, you must show all calculations using the method of undetermined coefficients, not merely quote a formula for the final result.

Answer:	$u(t) = -\frac{1}{15}\cos(4t) + \frac{1}{15}\cos(t)$

$mg = 32 \Rightarrow m \cdot 32 = 32 \Rightarrow \underline{m = 1}$ slug

$32 = mg = kL = k \cdot 2 \Rightarrow \underline{k = 16}$

$\gamma = 0$ (no damping)

Initial conditions: $u(0) = 0$, $u'(0) = 0$. So the IVP is

$$u'' + 16u = \cos t, \quad u(0) = 0 = u'(0).$$

Char eqn: $r^2 + 16 = 0 \Rightarrow r = \pm 4i$; $\lambda = 0$, $\mu = 4$

$$u_c(t) = c_1 \cos(4t) + c_2 \sin(4t)$$

Undet'd coeff to get u_p:

$$u_p(t) = A\cos t + B\sin t$$

$$u_p'(t) = -A\sin t + B\cos t$$

$$u_p''(t) = -A\cos t - B\sin t$$

$$u_p'' + 16u_p = -A\cos t - B\sin t + 16(A\cos t + B\sin t)$$
$$= 15A\cos t + 15B\sin t = \cos t$$

$\therefore$ $\underline{A = \frac{1}{15},\ B = 0}$, $\quad u(t) = c_1\cos(4t) + c_2\sin(4t) + \frac{1}{15}\cos t$

$0 = u(0) = c_1 + \frac{1}{15} \Rightarrow \underline{c_1 = -\frac{1}{15}}$

$u'(t) = -4c_1\sin(4t) + 4c_2\cos(4t) - \frac{1}{15}\sin t$

$0 = u'(0) = 4c_2 \Rightarrow \underline{c_2 = 0}$