Name: KEY

Math 307D Second Midterm Exam
March 1, 2013

Instructions: There are six problems, with the value of each problem indicated, for a total of 100 points. You are allowed the use of one page of handwritten notes, front and back, on standard size paper. You are also allowed use of a scientific calculator (but graphing calculators and other calculational devices are not allowed).

- Work the problems in the space provided. If you need more space, use the back of the page, and clearly indicate that you are doing so.
- Neatness counts! A well-organized solution, even with mistakes, will get more partial credit than a haphazard collection of unrelated calculations.
- Put the answer you want considered in the $\boxed{\textbf{Box}}$ provided.
- You **MUST** show all your work and reasoning to receive credit. If in doubt, ask for clarification.
- Turn off all cell phones and pagers.

Problem 1	20 points	
Problem 2	10 points	
Problem 3	10 points	
Problem 4	20 points	
Problem 5	20 points	
Problem 6	20 points	
Total	100 points	

1. (20 points). Find the solution to the initial value problem

$$y'' + 2y' - 3y = 0, \qquad y(0) = 0, \quad y'(0) = -4.$$

Answer: $\boxed{y(t) = e^{-3t} - e^{t}}$

The char eqn is $r^2 + 2r - 3 = 0$

or $(r+3)(r-1) = 0$, so $r = -3$ or $r = 1$

The general solution is

$$y(t) = Ae^{-3t} + Be^{t}$$

Using the initial conditions:

$$0 = y(0) = A + B$$

$$y'(t) = -3Ae^{-3t} + Be^{t}$$

$$-4 = y'(0) = -3A + B$$

Eliminate B by subtracting:

$$\begin{array}{r} 0 = A + B \\ -(-4 = -3A + B) \\ \hline 4 = 4A \end{array} \Rightarrow A = 1, \text{ so } B = -1$$

and $y(t) = e^{-3t} - e^{t}$

2. (10 points) Put the complex number

$$e^{1+i}$$

into rectangular form $a + ib$

Answer: $\boxed{e^{1+i} = (e\cos 1) + i\,(e\sin 1)}$

$$e^{1+i} = e \cdot e^{i}$$

$$= e \cdot (\cos 1 + i \sin 1)$$

$$= \underbrace{e \cdot \cos 1}_{\substack{\parallel \\ a}} + i\,\underbrace{(e \sin 1)}_{\substack{\parallel \\ b}}$$

3. (10 points) Find the general solution to

$$y'' + 6y' + 9y = 0.$$

Answer: $\boxed{y(t) = Ae^{-3t} + Bte^{-3t}}$

The char equ is $r^2 + 6r + 9 = 0$

or $(r+3)^2 = 0$

so $r = -3$ is a repeated root.

$\therefore$ general solution is

$$y(t) = Ae^{-3t} + Bte^{-3t}$$

4. (20 points) Find the general solution to

$$y'' - 5y' + 6y = 6t^2 + e^t.$$

Answer: $\boxed{y(t) = Ae^{2t} + Be^{3t} + t^2 + \frac{5}{3}t + \frac{19}{18} + \frac{1}{2}e^t}$

First find y_c for the homogeneous equation:

$$y'' - 5y' + 6y = 0$$

Char eqn is $r^2 - 5r + 6 = 0$, or $(r-2)(r-3) = 0$, so $r = 2$ and $r = 3$ are roots. $\therefore$

$$y_c(t) = Ae^{2t} + Be^{3t}.$$

The right side is $g_1(t) + g_2(t)$, $g_1(t) = 6t^2$, $g_2(t) = e^t$ and use undet'd coeff's for each piece:

y_{P_1} has form $Ct^2 + Dt + E$

$$y_{P_1}' = 2Ct + D, \quad y_{P_1}'' = 2C$$

$$y_{P_1}'' - 5y_{P_1}' + 6y_{P_1} = 2C - 5(2Ct + D) + 6(Ct^2 + Dt + E)$$

$$\text{so} \quad 6t^2 = (6C)t^2 + (6D - 10C)t + (6E - 5D + 2C)$$

$$\Rightarrow C = 1, \quad 6D - 10 = 0 \Rightarrow D = \frac{10}{6} = \frac{5}{3}$$

$$6E = 5D - 2C = \frac{25}{3} - 2 = \frac{19}{3} \Rightarrow E = \frac{19}{18}$$

$$\text{so } y_{P_1} = t^2 + \frac{5}{3}t + \frac{19}{18}$$

y_{P_2} has form Fe^t, so

$$e^t = Fe^t - 5Fe^t + 6Fe^t = 2Fe^t \Rightarrow F = \frac{1}{2}$$

$$y_{P_2} = \frac{1}{2}e^t$$

5. (20 points) The equation

$$t^2y'' - ty' - 3y = 0$$

has $y_1(t) = t^3$ as one solution. Find a second linearly independent solution. Be sure to check your answer!

Answer: $\boxed{y_2(t) = \frac{1}{t}}$

Let $y_2(t) = v(t)\,y_1(t) = t^3 v$

$y_2' = t^3 v' + 3t^2 v$

$y_2'' = t^3 v'' + 6t^2 v' + 6t v$

$\therefore\ t^2 y_2'' - t y_2' - 3y_2 = t^2\left[t^3 v'' + 6t^2 v' + 6tv\right]$
$\quad - t\left[t^3 v' + 3t^2 v\right]$
$\quad - 3t^3 v$

$= t^5 v'' + \left[6t^4 - t^4\right]v' + \cancelto{0}{\left(6t^3 - 3t^3 - 3t^3\right)} v = 0$

or $\quad t^5 v'' + 5t^4 v' = 0$

or $\quad v'' + \left(\frac{5}{t}\right) v' = 0$

Integrating factor is $\mu(t) = e^{\int \frac{5}{t}dt} = e^{5\ln t} = t^5$

So $\quad \left[t^5 v'\right]' = 0$

$\Rightarrow \quad t^5 v' = c$

$v' = \frac{c}{t^5} = ct^{-5}$

$\Rightarrow \quad v = \int ct^{-5}dt = c_1 t^{-4}$, so use $v = t^{-4}$

So $\quad y_2(t) = t^3 \cdot t^{-4} = \frac{1}{t} = t^{-1}$

6. (20 points) A mass weighing 16 lbs stretches a spring 2 ft. There is no damping. The mass is suddenly set in motion from its equilibrium position by giving it a downward velocity of 3ft/sec. Determine the displacement $u(t)$ from the equilibrium position as a function of the time t.

Answer: $\boxed{u(t) = \frac{3}{4}\sin(4t)}$

First determine m:

$$16\,lb = m \cdot g = m\,(32\,ft/sec^2) \Rightarrow m = \tfrac{1}{2}\text{ slug.}$$

Next determine k:

$$mg = kL \Rightarrow 16\,lb = k \cdot 2\,ft \Rightarrow k = 8$$

There is no damping, so $\gamma = 0$

There is no external force: $F(t) = 0$.

So the equation is

$$\tfrac{1}{2}u'' + 8u = 0, \quad \text{or} \quad u'' + 16u = 0$$

Char eqn is $r^2 + 16 = 0 \Rightarrow r = \pm 4i$, so general solution is

$$u(t) = A\cos(4t) + B\sin(4t)$$

The initial conditions are $u(0) = 0$, $u'(0) = 3$

$$0 = u(0) = A$$

$$u'(t) = -4A\sin(4t) + 4B\cos(4t)$$

$$3 = u'(0) = 4B \Rightarrow B = \tfrac{3}{4}.$$

So

$$u(t) = \tfrac{3}{4}\sin(4t)$$