# Math-Stat-491-Fall2014-Notes-VI

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December 3, 2014

## Poisson processes

## 1 Two equivalent definitions

**Definition 1.1.** *(Poisson process)*
*A* $1-$ *dimensional homogeneous Poisson process is a function* $N(\cdot)$ *on* $\Re_+$, *whose value on* $s \in \Re_+$ *is an integer valued random variable* $N(s)$ *satisfying the following conditions:*

1. $N(0) = 0$,

2. $N(s)$ *obeys the Poisson distribution:*

$$P(N(s) = n) = e^{-\lambda s}(\lambda s)^n/n!.$$

3. $N(s+t) - N(s)$, *has the same distribution as* $N(t)$ *and is independent of* $N(r), 0 \leq r \leq s$.

A good way of visualising a homogeneous Poisson process is to imagine random points marked on the nonnegative real line satisfying the following:

1. The probability of $n$ points occurring in the interval $[0, s]$ is given by the above Poisson distribution.

2. The probability of $k$ points occurring in the interval $[s, s+t]$ is the same as that occurring in the interval $[0, t]$.

3. If $[a, b), (c, d]$ are disjoint intervals the events, $k$ points occurring in $[a, b)$ and $m$ points occurring in $(c, d]$, are independent.

A second way of imagining a Poisson process is in terms of arrival in a queue which does not reduce in length because the person at the window is infinitely slow. The arrival takes place at random times and $N(s)$ counts the number of persons who have arrived in the interval $[0, s]$. If we take the time between arrivals to be exponentially distributed and independent, then $N(s)$ can be shown to be a Poisson process. Also every one dimensional Poisson process can be captured by this description.

The formal definition which can be shown to be equivalent to the earlier definition of Poisson process is as follows:

**Definition 1.2.** *(Poisson process)*
*Let $\tau_1, \tau_2, \cdots$ be independent $exp(\lambda)$ random variables. Let $T_n \equiv \tau_1 + \cdots \tau_n$. We define the Poisson process $N(s)$ as $N(s) \equiv max\{n : T_n \leq s\}$.*

(This means that $T_n \leq s$ and $T_{n+1} > s$.)

We will now look at consequences of Definition 1.2 of a Poisson process.

**Theorem 1.1.** $P(N(s) = n) = e^{-\lambda s}(\lambda s)^n/n!$, *i.e., $N(s)$ has a Poisson distribution with mean $\lambda s$.*

Proof: $N(s) = n$ can be thought of as the disjoint union of the following events: $T_n$ lies in the interval $(t_i, t_i + \Delta(t))$ and $\tau_{n+1} > s - t_i$, for $i = 0, \cdots (s/\Delta t)$.
(Note that the interval $[0, s]$ has been broken up into disjoint sub intervals of width $\Delta t$.)
Now $P(T_n \in (t_i, t_i + \Delta(t)) = f_{T_n}(t)\Delta t$, where $f_{T_n}(t)$ is the density of the Poisson distribution. So probability $\{T_n$ lies in the interval $(t_i, t_i + \Delta(t))$ *and* $\tau_{n+1} > s - t_i\} = f_{T_n}(t)P(\tau_{n+1} > s - t_i)\Delta t$. As $\Delta t \to 0$, therefore

$$P(N(s) = n) = \int_0^s f_{T_n}(t)P(\tau_{n+1} > s - t)dt.$$

We show in the next section that $f_{T_n}(t) = \lambda e^{-\lambda t}[(\lambda t)^{n-1}/(n-1)!] for\ t \geq 0$ and we know $P(\tau_{n+1} > s - t) = e^{s-t}$. So

$$P(N(s) = n) = \int_0^s \lambda e^{-\lambda t}[(\lambda t)^{n-1}/(n-1)!]e^{-\lambda(s-t)}dt$$

$$= [\lambda^n/(n-1)!]e^{-\lambda s} \int_0^s t^{n-1}dt = e^{-\lambda s}(\lambda s)^n/n!.$$

# 2 Axiomatic approach to Poisson processes

The previous section describes the Poisson process in a way which strongly brings in properties of the positive real line. While this is convenient for the 1 dimensional process, it masks its very general nature - that it can be viewed in terms of random occurrence of points in regions of $n$ dimensions. We therefore give an axiomatic characterization of the Poisson process which is valid in $n$ dimensions. For easy readability we confine ourselves to the homogeneous process. The axioms are stated for the $n$ dimensional process. But the reader may, for a first reading, take $n = 1$.

**Definition 2.1.** *Let $S$ be a region in $\Re^n$, such that $S = \bigcup_{i=1}^{\infty} S_i$, where the volume $\int_{S_i} dx$ of each region $S_i$ is finite. Let $\lambda \geq 0$. (The volume is a measure assigned to a $\sigma-$ algebra of regions, 'dx' is the infinitesimal volume.) A Poisson process of intensity $\lambda$ on $S$ satisfies the folowing:*

1. *The probability $p_k(\mu)$ that a region $T \subset S$ contains $k$ random points depends only on its mass $\mu = \int_T \lambda dx$. If $\mu(T)$ is infinite then all $p_k(\mu)$ are zero. The mass of a single point is taken to be zero.*

2. *The numbers of points in disjoint regions are independent.*

3. *The first two probabilities $p_0(\mu), p_1(\mu)$ have the asymptotic values as $\mu$ tends to zero,*

$$p_0(\mu) = 1 - \mu + o(\mu)$$

$$p_1(\mu) = \mu + o(\mu).$$

*Therefore for all $k > 1$, we have $p_k(\mu) = o(\mu)$.*

Note that a function $f(x)$ is $o(x)$ if $lim_{x \to 0} f(x)/x = 0$.

# 3 Consequences of the Poisson process axioms

## 3.1 Number of points follows Poisson distribution

A consequence of the axioms is the following

**Theorem 3.1.** *If the conditions in Definition 2.1 are satisfied, then the number of random points in a region of mass $\mu$ has the Poisson distribution*

$$p_k(\mu) = e^{-\mu}(\mu)^k/k!.$$

Proof: Let $\mu, d\mu$ be the masses of two non overlapping regions. By making the region as small as we please (this is possible because the space is $\Re^n$), we can make $d\mu$ as small as we please. The probability of there being no points in the union of the disjoint regions is, using the independence and asymptotic conditions,

$$p_0(\mu + d\mu) = p_0(\mu)p_0(d\mu) = p_0(\mu)(1 - d\mu + o(d\mu)).$$

Rearranging terms we get

$$(p_0(\mu + d\mu) - p_0(\mu))/d\mu = -p_0(\mu) + o(1).$$

In the limit as $d\mu$ tends to zero, we get,

$$d/d\mu[p_0(\mu)] = -p_0(\mu).$$

From the asymptotic conditions we infer $p_0(0) = 1$. Therefore, the solution to the above differential equation is

$$p_0(\mu) = e^{-\mu}.$$

For $k \geq 1$, we get from the postulates,

$$p_k(\mu + d\mu) = p_k(\mu)p_0(d\mu) + p_{k-1}(\mu)p_1(d\mu) + \cdots + p_0(\mu)p_k(d\mu)$$

$$= p_k(\mu)p_0(d\mu) + p_{k-1}(\mu)p_1(d\mu) + o(d\mu).$$

Rearranging terms we get

$$(p_k(\mu + d\mu) - p_k(\mu))/d\mu = -p_k(\mu) + p_{k-1}(\mu) + o(1).$$

Taking limits we get the differential equation

$$d/d\mu[p_k(\mu)] = -p_k(\mu) + p_{k-1}(\mu),$$

with initial condition $p_k(0) = 0$.

To solve these differential equations we make the transformation,

$$q_k(\mu) = p_k(\mu)e^{\mu}.$$

We have,

$$d/d\mu[q_k(\mu)] = d/d\mu[p_k(\mu)]e^{\mu} + p_k(\mu)e^{\mu} = e^{\mu}[-p_k(\mu) + p_{k-1}(\mu) + p_k(\mu)] = q_{k-1}(\mu).$$

Thus,

$$q_k(\mu) = \int_0^{\mu} q_{k-1}(\alpha)d\alpha.$$

We therefore get

$$q_0(\mu) = 1, p_0(\mu) = e^{-\mu}$$

$$q_1(\mu) = \mu, p_1(\mu) = e^{-\mu}\mu$$

$$q_2(\mu) = \mu^2/2, p_2(\mu) = e^{-\mu}\mu^2/2$$

$$\cdots,$$

$$q_r(\mu) = \mu^r/r!, p_r(\mu) = e^{-\mu}\mu^r/r!.$$

### 3.2 Superposition

Suppose we have random points assigned to subregions of $S$ according to two independent processes which have intensities $\lambda_1, \lambda_2$ respectively. The superposition of these two processes on $S$, would satisfy all the axioms but with mass of a region $T \subset S$ being given by $\mu_{12}(T) = \int_T [\lambda_1 + \lambda_2]dx$. We therefore would have a Poisson process with intensity $\lambda_1 + \lambda_2$.

### 3.3 Coloring

Consider the following process: we assign random points to subregions of $S$ according to a Poisson process of intensity $\lambda$, after that we color them as belonging to one of $k-$ colors with the probability of being colored by $i^{th}$ color being $p(i)$. The axioms would be satisfied again but with $n \times p(i)$ points in a region assigned to color $i$, when the original process assigned it $n$ points. It follows that the $i^{th}$ color process is Poisson with intensity $\lambda p(i)$.

### 3.4 Conditional distribution of arrival times

Suppose, in the interval $[0, t]$, we know that exactly one event of a Poisson process $N(t)$ has taken place. What is the probability that it lies in the interval $[s, r]$?

We have

$$Pr\{X_1 \in [s, r] \mid N(t) = 1\} = Pr\{X_1 \in [s, r], N(t) = 1\}/Pr\{N(t) = 1\}$$

$$Pr\{X_1 \in [s, r], 0 \ events \ in[0, s], 0 \ events \ in(s, t]\}/Pr\{N(t) = 1\}$$

$$= \lambda(r - s)e^{-\lambda(r-s)}e^{-\lambda s}e^{-\lambda(t-r)}/\lambda te^{-\lambda t} = (r - s)/t.$$

Thus under the condition that exactly 1 event has occurred, the probability that the event lies in an interval is simply the ratio of the length of the interval to the length of the overall interval, i.e., according to the uniform distribution.

Next given that $N(t) = n$, we will compute the density of the distribution of the arrrival times $S_1, \cdots, S_n$.

Let $0 < t_1 < \cdots < t_{n+1} = t$ and let $h_i$ be small enough so that $t_i + h_i < t_{i+1}, i = 1, \cdots, n$. We then have

$$Pr\{t_i \leq S_i \leq t_i + h_i, i = 1, 2 \cdots n \mid N(t) = n\}$$

$$= Pr\{exactly\ one\ event\ in\ [t_i, t_i + h_i], i = 1, 2 \cdots n, no\ events\ elsewhere\ in\ [0, t]\}/Pr\{N(t) = n\}$$

$$= [\lambda h_1 e^{-\lambda h_1} \cdots \lambda h_n e^{-\lambda h_n} e^{-\lambda(t - h_1 - h_2 - \cdots - h_n)}]/[e^{-\lambda t}(\lambda t)^n/n!]$$

$$= [n!/t^n][h_1 h_2 \cdots h_n].$$

Therefore, the conditional density $f(t_1, \cdots t_n)$ of $S_1, \cdots, S_n$ at arrivals $t_1, \cdots t_n$,

$$= \lim_{h_i \to 0} [Pr\{t_i \leq S_i \leq t_i + h_i, i = 1, 2 \cdots n \mid N(t) = n\}/h_1 h_2 \cdots h_n] = n!/t^n.$$

# 4 Exponential distribution

Since Poisson processes can be defined in terms of the exponential distribution, we list the properties of the latter which give Poisson processes their characteristics.

1. We say $T = exp(\lambda)$, if $P(T \leq t) = 1 - e^{-\lambda t}, \forall t \geq 0$.

2. If $T = exp(\lambda)$, then its density function

$$f_T(t) = \lambda e^{-\lambda t}, t \geq 0; f_T(t) = 0, t < 0.$$

$$E[T] = 1/\lambda; E[T^2] = 2/(\lambda)^2; E[T^n] = n!/(\lambda)^n;$$

$$Var[T] = 1/(\lambda)^2.$$

3. (**memory less**)
   $P(T > t + s \mid T > t) = P(T > s)$
   Proof: $P(T > t + s \mid T > t) = P(T > t + s; T > t)/P(T > s) = P(T > t + s)/P(T > s)$

$$= e^{-\lambda(t+s)}/e^{-\lambda t} = e^{-\lambda s} = P(T > s).$$

4. ($P(min(S, T) > t), S, T$ independent)
   Let $S = exp(\mu), T = exp(\lambda)$. Then, if they are also independent,

$$P(min(S, T) > t) = P(S > t, T > t) = P(S > t)P(T > t) = e^{-\mu t}e^{-\lambda t} = e^{-(\lambda + \mu)t}.$$

   More generally given independent random variables $T_i = exp(\lambda_i)$, the probability that the minimum of them is greater than $t$ is $e^{-\Sigma_i \lambda_i t}$.

5. $(P(S<T), S, T$ independent)

Probability that $S<T$ in an interval $(t, t+\Delta s)$ is the probability that $S$ lies in that interval and that $T>t+\Delta s$. This is $(f_S(s)\Delta s) \times P(T>s)$. We divide $[0,\infty]$ into such intervals and sum this probability over all of them letting $\Delta s$ tend to 0.

$$P(S<T) = \int_0^\infty f_S(s)P(T>s)ds$$

$$= \int_0^\infty f_S(s)e^{-\mu s}ds$$

$$= \int_0^\infty \lambda e^{-\lambda s}e^{-\mu s}ds$$

$$= \lambda/(\lambda+\mu).$$

More generally, the probability that among the independent random variables $T_i = exp(\lambda_i), T_j$ is the minimum, is

$$\lambda_j/\Sigma_i\lambda_i.$$

6. If $\tau_1, \tau_2, \cdots$ are independent $exp(\lambda)$ random variables, then the sum $T_n = \tau_1+\tau_2+\cdots+\tau_n$ has a $gamma(n,\lambda)$ distribution, its density function being

$$f_{T_n}(t) = \lambda e^{-\lambda t}[(\lambda t)^{n-1}/(n-1)!] for\ t \geq 0\ \ and\ \ 0\ \ for\ \ t<0.$$

Proof:

The proof is by induction. For $n=1$, we have $f_{T_1}(t) = \lambda e^{-\lambda t} = \lambda e^{-\lambda t}[(\lambda t)^0/(0)!]$, so the result is true.

Suppose it is true for $n$. Recall that if $X, Y$ are independent and take value 0 for $t<0$, the density $f_{X+Y}(t)$ at $t$ is given by $\int_0^t f_X(s)f_Y(t-s)ds$.

In the present case $T_n$ and $\tau_{n+1}$ are independent. Therefore

$$f_{T_{n+1}}(t) = \int_0^t \lambda e^{-\lambda s}[(\lambda s)^{n-1}/(n-1)!] \times \lambda e^{-\lambda(t-s)}ds = e^{-\lambda t}\lambda^{n+1}\int_0^t s^{n-1}/(n-1)!ds = \lambda e^{-\lambda t}\lambda^n t^n/n!,$$

proving the result.