Name: KEY

Math 307D First Midterm Exam

February 1, 2013

Instructions: There are five problems, each worth 20 points, for a total of 100 points. You are allowed the use of a scientific calculator (but graphing calculators and other calculational devices are **NOT ALLOWED**). You are also allowed the use of one page of handwritten notes, front and back, standard sized paper.

- Work the problems in the space provided. If you need more space, use the back of the page, and clearly indicate that you are doing so.
- Neatness counts! A well-organized solution, even with mistakes, will get more partial credit than a haphazard collection of unrelated calculations.
- Put the answer you want considered in the $\boxed{\textbf{Box}}$ provided.
- You **MUST** show all your work and reasoning to receive credit. If in doubt, ask for clarification.
- Turn off all cell phones and pagers.

Problem 1	20 points	
Problem 2	20 points	
Problem 3	20 points	
Problem 4	20 points	
Problem 5	20 points	
Total	100 points	

1. (20 points). Find the solution to the initial value problem

$$\frac{dy}{dx} = \frac{4x^3}{(2+x^4)y}, \quad y(0) = -1.$$

Make sure you write your solution for y as an *explicit* function of x.

Answer: $\boxed{y = -\sqrt{2\ln(x^4+2)+1-2\ln 2}}$

Separable:

$$y\,dy = \frac{4x^3}{2+x^4}\,dx$$

$$\int y\,dy = \int \frac{4x^3}{2+x^4}\,dx$$

$$\frac{y^2}{2} = \ln(x^4+2) + C$$

$y(0)=-1 \Rightarrow$ $$\frac{(-1)^2}{2} = \ln(2) + C \Rightarrow C = \frac{1}{2} - \ln 2$$

$$\frac{y^2}{2} = \ln(x^4+2) + \frac{1}{2} - \ln 2$$

$$y^2 = 2\ln(x^4+2) + 1 - 2\ln 2$$

$$y = \pm\sqrt{2\ln(x^4+2)+1-2\ln 2}$$

Choose $-$ sign since $y(0)=-1$

2. (20 points). Solve the initial value problem

$$t\,y' + 2y = e^{3t}, \quad y(1) = 0.$$

Make sure you write your solution for y as an *explicit* function of x.

Answer: $\boxed{y = \frac{1}{t^2}\left[\frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} - \frac{2}{9}e^{3}\right]}$

First put into standard form:

$$y' + \left(\frac{2}{t}\right)y = \frac{e^{3t}}{t}$$

Integrating factor is $\mu(t) = e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2.$

Get

$$(t^2 y)' = t^2\left(\frac{e^{3t}}{t}\right) = te^{3t}$$

$$t^2 y = \int \underbrace{t}_{u}\,\underbrace{e^{3t}dt}_{dv} \stackrel{(\text{parts})}{=} t\left(\frac{e^{3t}}{3}\right) - \int \frac{e^{3t}}{3}dt$$

$du = dt \quad v = \frac{e^{3t}}{3}$

$$= \frac{1}{3}te^{3t} - \frac{e^{3t}}{9} + C$$

$$y(1) = 0 \Rightarrow \quad 0 = \frac{1}{3}e^{3} - \frac{e^{3}}{9} + C \Rightarrow C = -\frac{2}{9}e^{3}$$

$$\Rightarrow \quad y = \frac{1}{t^2}\left[\frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} - \frac{2}{9}e^{3}\right]$$

3. (20 points). Water is pouring into a tank at a rate proportional to the **square root** of the amount of water present in the tank at that instant. At time $t = 0$ min there is 1 gallon of water in the tank, and at time $t = 1$ min there are 4 gallons of water in the tank. How many gallons of water are in the tank at time $t = 4$ min?

Answer: $\boxed{25 \text{ gal}}$

Let $W(t)$ = amount of water at time t min

$$\frac{dW}{dt} \propto \sqrt{W} \Rightarrow \frac{dW}{dt} = k\sqrt{W}, \quad k > 0$$

Separable:

$$\frac{dW}{\sqrt{W}} = k\,dt$$

$$\int \frac{dW}{\sqrt{W}} = \int k\,dt \Rightarrow 2\sqrt{W} = kt + C$$

$$W(0) = 1 \Rightarrow \quad 2 = C$$

$$W(1) = 4 \Rightarrow \quad 2\sqrt{4} = k + 2 \quad \Rightarrow k = 2$$

$$2\sqrt{W} = 2t + 2 \Rightarrow \sqrt{W} = t + 1$$

$$W = (t+1)^2$$

$$W(4) = 5^2 = 25 \text{ gal}$$

4. (20 points). Consider the autonomous differential equation

$$y' = y^3 - 2y.$$

(a) Find all equilibrium solutions to the equation.

Answer: $y = 0, \; y = \sqrt{2}, \; y = -\sqrt{2}$

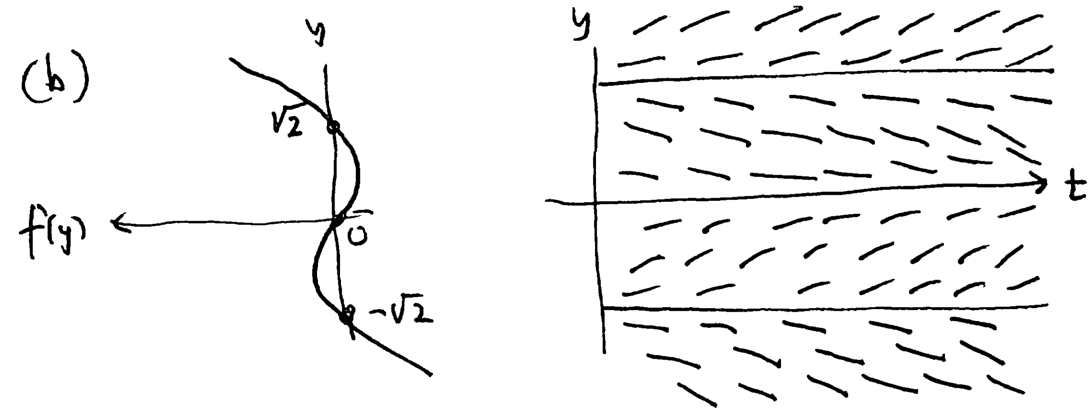
(b) In the space below, sketch the direction field for the equation so the all equilibrium solutions appear in your sketch. Be sure to label your axes.

(c) For each equilibrium solution in part (a) say whether it is stable, unstable, or semistable, briefly justifying each answer.

Answer: $y=0$ is stable
$y=\sqrt{2}$ and $y=-\sqrt{2}$ are unstable

(a) Plot $f(y) = y^3 - 2y$. Equilibrium solutions where $f(y) = 0 \Rightarrow y(y^2-2) = 0 \Rightarrow y = 0, \sqrt{2}, -\sqrt{2}$.

(b)



(c) $y=0$ is stable (nearby solutions converge)
$y = \sqrt{2}$ is unstable (nearby solutions diverge)
$y = -\sqrt{2}$ is unstable (")

5. Consider the initial value problem

$$\frac{dy}{dt} = t, \quad y(0) = 1.$$

(a) Use Euler's method, with step size $h = 0.1$ and starting at $t_0 = 0$, to approximate the value of the solution at $t = 0.4$.

Answer: $\boxed{1.06}$

(b) As a check, solve the equation to get the exact value of the solution at $t = 0.4$. In the box below put both the solution $y(t)$ and the exact value $y(0.4)$.

Answer: $\boxed{y = \frac{t^2}{2} + 1, \quad y(0.4) = 1.08}$

(a) $f(t, y) = t$

n	t_n	y_n
0	0.0	1.0
1	0.1	1.0
2	0.2	1.01
3	0.3	1.03
4	0.4	1.06

$$y_1 = y_0 + f(t_0, y_0)h$$
$$= 1.0 + 0(0.1) = 1.0$$

$$y_2 = y_1 + f(t_1, y_1)h$$
$$= 1.0 + (0.1)(0.1) = 1.01$$

$$y_3 = 1.01 + (0.2)(0.1) = 1.03$$

$$y_4 = 1.03 + (0.3)(0.1) = 1.06$$

(b) Separable: $dy = t\,dt \Rightarrow y = \frac{t^2}{2} + C$

$y(0) = 1 \Rightarrow C = 1$, so $y = \frac{t^2}{2} + 1$

$$y(0.4) = 1 + \frac{(0.4)^2}{2} = 1.08$$