Heat Flow and a Faster Algorithm to Compute the Surface Area of a Convex Body*

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Abstract

We draw on the observation that the amount of heat diffusing outside of a heated body in a short period of time is proportional to its surface area, to design a simple algorithm for approximating the surface area of a convex body given by a membership oracle. Our method has a complexity of $O^*(n^4)$, where n is the dimension, compared to $O^*(n^8)$ for the previous best algorithm. We show that our complexity cannot be improved given the current state-of-the-art in volume estimation.

1 Introduction

An important class of algorithmic questions centers around estimating geometric invariants of convex bodies. Arguably, the most basic invariant is the volume. It can be shown (see [6], [1]) that any deterministic algorithm that approximates the volume of a convex body within a constant factor in $\mathbb{R}^n$, needs time exponential in the dimension n. Remarkably, randomized algorithms turn out to be more powerful. In their pathbreaking paper [4] Dyer, Frieze and Kannan gave the first randomized polynomial time algorithm to approximate the volume of a convex body to arbitrary accuracy. Since then a considerable body of work has been devoted to improving the complexity of volume computation culminating with the recent best of $O^*(n^4)$ due to Lovász and Vempala [15].

Another fundamental geometric invariant associated with a convex body is surface area. Estimating the surface area was mentioned as an open problem by Grötschel, Lovász, and Schrijver in 1988 [13]. Dyer, Gritzmann and Hufnagel [5] showed in 1998 that it could be solved in randomized polynomial time. The primary focus of their paper was to establish that the computation of surface area and certain other mixed volumes was possible in randomized polynomial time, and they assumed access to oracles for δ-neighborhoods of the convex body. They did not discuss the complexity of their algorithm given only a membership oracle for the convex body. Below, we indicate an $O^*(n^8)$ analysis of their algorithm in terms of the more restricted queries.

In this paper we develop a new technique for estimating volumes of boundaries based on ideas from heat propagation. The underlying intuition is that the amount of heat escaping from a hot object in a small interval of time is proportional to the surface area. This idea can also be used to estimate the volume of a hypersurface possessing a tubular neighborhood of a given thickness. This corresponds to the hypersurface having a given "reach", where the reach of the hypersurface is defined as the largest number t such that any point at a distance atmost t from the hypersurface has a unique nearest point on the hypersurface. Details of this procedure may be found in [19].

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It turns out that this intuition lends itself to an efficient randomized algorithm for computing surface areas of convex bodies, given by a membership oracle. In this paper we describe the algorithm and the analysis of the algorithm, proving a complexity bound of $O^*(n^4)$. The $O^*(\cdot)$ notation hides the polynomial dependence on the relative error ϵ, and poly-logarithmic factors in the parameters of the problem. Since, as will be shown below, surface area estimation is at least as hard as volume approximation, this bound is the best possible, given the current state-of-the-art in volume estimation.

We note that this bound cannot be obtained using methods previously proposed in [5] due to a bottleneck in their approach. The method in [5] exploited the fact that $\mathrm{vol}(K+B\delta)$ is a polynomial in δ, where $B\delta$ is a ball of radius δ and the Minkowski sum $K+B\delta$ corresponds to the set of points within a distance δ of K. The surface area is the coefficient of the linear term, which their method estimated by interpolation. However, in a natural setting, we only have access to a membership oracle for K, but not for $K+B\delta$. Therefore a membership oracle for $K+B\delta$ has to be constructed, which as far as we can see, requires solving a quadratic programming problem on a convex set. Given access only to a *membership oracle*, the best algorithm to handle this task is due to Lovász and Vempala, and makes $O^*(n^8)$ oracle calls [16], which gives a bound on the complexity of the algorithm in [5] that is $O^*(n^8)$.

Even with a stronger *separation oracle* the complexity of the method in [5] for computing mixed volumes (and hence, in particular, surface area) is $O^*(n^5)$, since the associated quadratic programming problem for computing the surface area requires $O^*(n)$ operations (see [22], [3].) On the other hand, the complexity of our method is $O^*(n^4)$ using only a *membership oracle*, matching the complexity of the volume computation of Lovász and Vempala [15].

1.1 Preliminaries

Throughout this paper, B will denote the unit n-dimensional ball, K will denote an n-dimensional convex body such that $rB \subseteq K \subseteq RB$. $S = \mathrm{vol}(\partial K)$ will denote the surface area of K and $V = \mathrm{vol}(K)$, its volume. We will denote the points in $\mathbb{R}^n$ by x, y, etc, and $n-$dimensional volume elements in integrals by $d\mathrm{vol}_x$, $d\mathrm{vol}_y$, etc.

Definition 1.1. *We denote the set of points within a distance δ of a convex body K (including K itself) by K_δ. This is called the outer parallel body of K and is convex. The set of points at a distance $\geq \delta$ to $\mathbb{R}^n \setminus K$ shall be denoted $K_{-\delta}$. This is called the inner parallel body of K and is convex. For any body K, we denote by ∂K, its boundary.*

Given $x \in K$, let H_x be a closest half-space to x not intersecting $K \setminus \partial K$. For $y \notin K$ define H_y to be the half-space farthest from y containing K.

Definition 1.2. *Let*

$$G_t(x,y) := \frac{\mathrm{e}^{-\|x-y\|^2/4t}}{(4\pi t)^{n/2}}.$$

and

$$F_t := \sqrt{\frac{\pi}{t}} \int_K \int_{\mathbb{R}^n \setminus K} G_t(x,y) d\mathrm{vol}_y d\mathrm{vol}_x.$$

Observation 1.1. *If $x \in \partial K_{-\delta}$ then the distance between x and H_x is δ. If $y \in \partial K_\delta$ then the distance between y and H_y is δ.*

Definition 1.3. *Let*

$$e(t,\delta) = \frac{1 - \mathrm{Erf}\left(\frac{\delta}{2\sqrt{t}}\right)}{2}$$

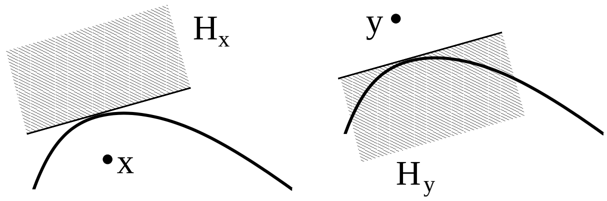

Figure 1: Points x and y and corresponding half-spaces H_x and H_y

where Erf *is the usual Gauss error function, defined by*

$$\mathrm{Erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2} dx.$$

Observation 1.2. *Let $x \in \partial K_{-\delta}$, and $y \in \partial K_\delta$. Then,*

$$\int_{H_y} G_t(z, y) d\mathrm{vol}_z = e(t, \delta)$$

and

$$\int_{H_x} G_t(z, x) d\mathrm{vol}_z = e(t, \delta)$$

The volume of K_δ is a polynomial in δ, given by the *Steiner formula* (see page 197, [21].)

$$\mathrm{vol}(K_\delta) = a_0 + \cdots + \binom{n}{i} a_i \delta^i + \cdots + a_n \delta^n.$$

The coefficients a_i satisfy the *Alexandrov-Fenchel inequalities* (see page 334, [21],) which state that the coefficients a_i are log-concave; i. e. $a_i^2 \geq a_{i-1} a_{i+1}$ for $1 \leq i \leq n-1$.

Definition 1.4. *The surface area* $\mathrm{vol}(\partial K)$ *of an arbitrary convex body K is defined as*

$$\lim_{\delta \to 0} \frac{\mathrm{vol} K_\delta - \mathrm{vol} K}{\delta}.$$

It follows from the *Steiner formula* that this limit exists and is finite. It is a consequence of Lemma 5.2 that the so defined surface area for an inner parallel body $\mathrm{vol}(\partial K_{-\delta})$ is a continuous function of δ. For an outer parallel body, the *Steiner formula* implies that $\mathrm{vol}(\partial K_\delta)$ is a polynomial in δ.

2 Hardness of estimating Surface Area

The problem of estimating the surface area of a convex body is at least as hard as that of estimating the volume.

Proposition 2.1. *If the surface area of any n-dimensional convex body K can be approximated in $O(n^\beta \mathsf{polylog}(\frac{nR}{\delta r}) \mathsf{poly}(\frac{1}{\epsilon}))$ time, the volume can also be approximated in $O(n^\beta \mathsf{polylog}(\frac{nR}{\delta r}) \mathsf{poly}(\frac{1}{\epsilon}))$ time, where δ is the probability that the relative error exceeds ϵ.*

The proof of this proposition will require the following lemma due to Wills (Lemma 1, [24])

Lemma 2.1. *Let K contain a ball of radius r_{in}. Then,*

$$\frac{r_{in}}{n} \leq \frac{V}{S}.$$

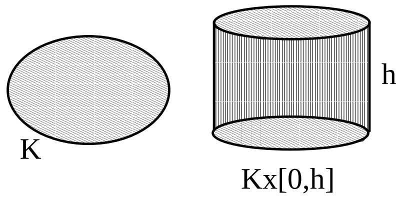

Figure 2: volume of the base and surface area of the cylinder.

Proof of Proposition 2.1. Given a convex body K, $rB \subseteq K \subseteq RB$ let $C(K,h) := K \times [0,h]$.

It is true for any $h > 0$ that

$$\text{vol } K = \frac{\text{vol } \partial C(K,h) - h\text{vol } \partial K}{2}.$$

By Lemma 2.1, it follows that for $h = \frac{\epsilon r}{n}$,

$$\frac{\text{vol}(K)}{h\text{vol}(\partial K)} \geq \frac{1}{\epsilon}.$$

Therefore,

$$\text{vol } K < \frac{\text{vol } \partial C(K,h)}{2} < (1+\epsilon)\text{vol } K.$$

Therefore algorithm that could approximate the surface area of $C(K,h)$ can also be used to approximate the volume of K, with the same polylogarithmic dependence on the relevant parameters. This completes the proof. □

Thus an efficient algorithm for surface area estimation would also lead to an almost equally efficient algorithm for estimating the volume.

3 Overview of the algorithm

Our approach provides an estimate for the isoperimetric ratio $\frac{S}{V}$. Using the fastest existing algorithm for volume approximation, we obtain a separate estimate for V. Multiplying these two estimates yields the surface area S.

The underlying intuition of our algorithm is that the heat diffuses from a hot body through its boundary, and the amount of heat escaping in a short period of time ought to be proportional to the surface area of the object. Recalling that a point source of heat diffuses at time t according to the Gaussian distribution $\frac{1}{(4\pi t)^{n/2}} e^{-\frac{\|x\|^2}{4t}}$ leads to the following informal description of the algorithm (see details in Section 4):

Step 1. Take $x_1, \ldots, x_N$ to be samples from the uniform distribution on K, where N will be chosen to be $O(n)$.

Step 2. For each x_i, let $y_i = x_i + v_i$, where v_i is sampled from the Gaussian distribution with density $\frac{1}{(4\pi t)^{n/2}} e^{-\frac{\|x\|^2}{4t}}$ for some appropriate value of t. Thus y_i is obtained from x_i by taking a random Gaussian step.

Step 3. Let $\hat{N}$ be the number of y's, which land outside of K. Note that $\frac{\hat{N}}{N}\sqrt{\frac{\pi}{t}}$ is an estimate for $\frac{S}{V}$.

Step 4. Using an existing algorithm, produce an estimate $\hat{V}$ for the volume. Estimate the surface area as $\hat{V}\frac{\hat{N}}{N}\sqrt{\frac{\pi}{t}}$.

We will show that that each of the Steps 1,3,4 can be done using at most $O^*(n^4)$ calls to the membership oracle[1]. Step 2, of course, does not require any calls to the oracle at all.

The main technical result of this paper is to show how to choose values of t and N, such that

$$(1-\epsilon)\left(\frac{S}{V}\sqrt{\frac{t}{\pi}}\right) < \frac{\hat{N}}{N} < (1+\epsilon)\left(\frac{S}{V}\sqrt{\frac{t}{\pi}}\right)$$

It is not known how to efficiently obtain independent random samples from the uniform distribution on K. We show how to relax this condition and use *almost independent* samples from a *nearly uniform* distribution instead, to derive these estimates.

We then apply certain results from [14] and [15], to generate $O\left(\frac{n}{\epsilon^3}\right)$ such samples making at most $O^*\left(\frac{n^4}{\epsilon^3}\right)$ oracle calls.

Putting these and some additional observations together, we obtain the following theorem which is the main result of this paper:

Theorem 3.1 (Main Theorem)**.** *The surface area of a convex body K, given by a membership oracle, and parameters r, R such that $rB \subseteq K \subseteq RB$ can be approximated to within a relative error of ϵ with probability $1-\delta$ using at most*

$$O\left(n^4 \log\frac{1}{\delta}\left(\frac{1}{\epsilon^2}\log^9\frac{n}{\epsilon} + \log^8 n \log\frac{R}{r} + \frac{1}{\epsilon^3}\log^7\left(\frac{n}{\epsilon}\right)\right)\right)$$

i. e. $O^*(n^4)$, oracle calls.

The number of arithmetic operations is $O^*(n^6)$, on numbers having polylogarithmic (in n) number of digits. This is the same as that for volume computation in [15].

4 Algorithm to compute the surface area

4.1 Notation and Preliminaries

Definition 4.1. *A body K is said to be in t-isotropic position if, for every unit vector u,*

$$\frac{1}{t} \le \int_K (u^T(x-\bar{x}))^2 d\mathrm{vol}_x \le t,$$

where $\bar{x}$ is the center of mass of K.

Let ρ be the uniform measure on convex body K. We call a random point x (which in our case is produced by a random walk based algorithm) ϵ-uniform if

$$\sup_{\text{measurable } A\subseteq K} \left|P(x\in A) - \rho(A)\right| \le \frac{\epsilon}{2}.$$

Two random variables will be called μ-independent if for any two Borel sets A and B in their ranges,

$$|P(X\in A, Y\in B) - P(X\in A)P(Y\in B)| \le \mu.$$

We define the $\mathcal{L}^2$ norm of a distribution η with respect to the uniform distribution (denoted ρ) on K to be

$$\|\eta\|_2 := \sqrt{\int_K \left(\frac{d\eta}{d\rho} - 1\right)^2 d\rho}.$$

[1]It is customary to count the number of oracle calls rather than the number of arithmetic steps in the volume literature, while measuring the complexity.

A consequence of the results on page 4 of [15], and Theorem 7.2 of [23] is the following. Suppose we are given a starting point that is sampled from an ϵ-uniform distribution possessing $\mathcal{L}^2$ norm bounded above by an absolute constant. Then, there is a random walk based algorithm that uses $O(n^3 \ln^7 \frac{n}{\epsilon\mu})$ oracle calls per point for generating N points $x_1, \dots, x_N$ that are ϵ-uniform with the further property that each pair is μ-independent from a convex body that is 2-isotropic. This fact plays a crucial role in allowing the surface area algorithm to have a complexity bounded by $O^*(n^4)$.

In the remainder of the paper, we frequently need to integrate by parts expressions of a certain kind. Therefore we collect some pertinent formulae below. Let $f(x)$ be any continuous function having no more than exponential growth, i. e.

$$\lim\sup_{x\to\infty} \frac{\ln f(x)}{x} < \infty. \tag{1}$$

Then,

$$\int f(\delta)e(t,\delta)d\delta = \left(\int f(\delta')d\delta'\right) e(t,\delta) - \int \left(\int f(\delta')d\delta'\right)\left(\frac{d}{d\delta}e(t,\delta)\right) d\delta. \tag{2}$$

Let

$$-\frac{d}{d\delta}e(t,\delta) =: N(t,\delta), \tag{3}$$

i. e.

$$N(t,\delta) = \frac{e^{-\frac{\delta^2}{4t}}}{\sqrt{4\pi t}}. \tag{4}$$

Thus from (2), we have the following.

$$\int_0^\infty f(\delta)e(t,\delta)d\delta = \left[\left(\int_0^\delta f(\delta')d\delta'\right) e(t,\delta)\right]\Bigg|_0^\infty + \int_0^\infty \left(\int_0^\delta f(\delta')d\delta'\right) N(t,\delta)d\delta. \tag{5}$$

The first term disappears, since $e(t,\delta)$ tends to 0 super-exponentially as $\delta \to \infty$. Therefore

$$\int_0^\infty f(\delta)e(t,\delta)d\delta = \int_0^\infty \left(\int_0^\delta f(\delta')d\delta'\right) N(t,\delta)d\delta. \tag{6}$$

4.2 Algorithm

We present an algorithm below that outputs an ϵ-approximation to the surface area of a convex body K with probability $> 3/4$. Running it $\lceil 36 \ln\left(\frac{2}{\delta}\right)\rceil$ times and taking the median of the outputs gives the result with a confidence $> 1 - \delta$.

Input: Convex body K, given by a membership oracle, and parameters r, R such that $rB \subseteq K \subseteq RB$ and an error parameter $\epsilon < 1$.
Output: An estimate $\hat{S}$, that with probability $> 3/4$ has a relative error of less than ϵ with respect to S.

Set $\epsilon' := \frac{\epsilon}{8}$, $\mu := \frac{\epsilon'^4}{2^{18}n^2}$, $N := \lceil \frac{2^{13}n}{\epsilon'^3}\rceil$.

Step 1. Run the a volume algorithm to obtain an estimate $\hat{V}$ of V that has a relative error ϵ' with probability $> \frac{15}{16}$.
Step 2 Generate a linear transformation T given by a symmetric positive-definite matrix such that TK is 2-isotropic with probability $> \frac{15}{16}$.

Step 3 Compute a lower bound r' to the smallest eigenvalue r_{opt} of $\frac{T^{-1}}{\sqrt{2}}$, that satisfies $\frac{2}{\sqrt{5}} r_{opt} < r' < r_{opt}$. Set $r_{in} := \max(r, r')$.
Step 4 Set $\sqrt{t} := \frac{\epsilon' r_{in}}{4n}$.
Step 5 Generate N random points $x_2, \dots, x_N$ from K, such that with probability 15/16, they are $\frac{\epsilon'^2}{64n}$-uniform and each pair $\{x_i, x_j\}$ for $1 \le i < j \le N$ is μ-independent.
Step 6 Generate N independent random samples $v_1, \dots, v_N$ from the spherically symmetric multivariate Gaussian distribution with mean $\vec{0}$ and variance $2nt$.
Step 7 Let $\hat{N} := |\{i | x_i + v_i \notin K\}|$ be the number of times $x_i + v_i$ lands outside of K.
Step 8 Output $\frac{\hat{N}}{N}\sqrt{\frac{\pi}{t}}\hat{V}$.

4.3 Analysis of the Run-time

Step 1 takes at most

$$O\left(\frac{n^4}{\epsilon^2}\log^9\frac{n}{\epsilon} + n^4\log^8 n \log\frac{R}{r}\right)$$

oracle calls, using the volume algorithm of Lovász and Vempala [15]. The number of steps in the computation is $O^*(n^6)$.
Step 2 Such a transformation is obtained during the execution of the volume algorithm from [15] for no additional cost.
Step 3 takes $O(n^3)$ steps of computation [20].
Step 4 takes $O(1)$ steps.
Step 5 takes

$$O\left(\frac{n^4}{\epsilon^3}\log^7\left(\frac{n}{\epsilon}\right)\right)$$

steps of computation (including oracle calls) once a point x_1 is obtained that is $\frac{\epsilon'^2}{64n}$-uniform, and has a distribution whose $\mathcal{L}^2$ norm is bounded above by a constant. Such a point can be obtained from the algorithm in step 1, for no additional cost up to constants. The cost mentioned in this step is incurred because we are required to generate $O(\frac{n}{\epsilon^3})$ random points given the initial random point x_1 and the time per point is $O(n^3 \ln^7 \frac{n}{\epsilon})$. This last fact follows from the complexity per point mentioned on page 4 [15], and theorems 7.1 and 7.2 of [23].
Step 6 and **Step 7** take $O\left(\frac{n^2}{\epsilon^3}\mathsf{polylog}\frac{n}{\epsilon\delta}\right)$ steps each, assuming that a sample from univariate Gaussian distribution can be obtained upto $O(\mathsf{polylog}(\frac{n}{\epsilon\delta}))$ digits in $O(\mathsf{polylog}(\frac{n}{\epsilon\delta}))$ steps.
Step 8 takes $O(1)$ steps. Finally, to obtain the approximation with a confidence $> 1-\delta$, this algorithm must be run $O\left(\log\left(\frac{1}{\delta}\right)\right)$ times. Therefore the overall cost in terms of oracle calls is

$$O\left(n^4 \log\frac{1}{\delta}\left(\frac{1}{\epsilon^2}\log^9\frac{n}{\epsilon} + \log^8 n \log\frac{R}{r} + \frac{1}{\epsilon^3}\log^7\left(\frac{n}{\epsilon}\right)\right)\right)$$

i.e. $O^*(n^4)$ oracle calls. The number of arithmetic operations is $O^*(n^6)$, on numbers with a polylogarithmic number of digits. This is the same as that for volume computation in [15]. That the estimate $\hat{S}$ of the surface area S output by the algorithm in **Step 8** lies in $[(1-\epsilon)S, (1+\epsilon)S]$ with probability at least $1-\delta$ is the content of Theorem 3.1.

5 Proof of the main theorem

In order to prove our main theorem, we require two propositions. The first, Proposition 5.1, states that F_t is a good approximation for the surface area S. The second, Proposition 5.2, states that the empirical quantity $\hat{S}$ (an estimate of S) computed by the surface area algorithm is likely to be an ϵ-approximation of F_t with probability $> 3/4$.

As in the surface area algorithm, let T be a linear transformation such that TK is 2-isotropic. Let r' be a lower bound to the smallest eigenvalue r_{opt} of $\frac{T^{-1}}{\sqrt{2}}$, that satisfies $\frac{2}{\sqrt{5}} r_{opt} < r' < r_{opt}$. Set $r_{in} := \max(r, r')$.

Proposition 5.1 (Relating surface area S to normalized heat flow F_t)**.** *Let $\sqrt{t} = \frac{\epsilon' r_{in}}{4n}$ and $\epsilon' < 1/2$. Then,*

$$(1 - \epsilon')S < F_t < (1 + \epsilon')S.$$

Proof of Proposition 5.1. We begin by lower bounding $\mathrm{vol}(\partial K_{-\delta})$ and upper bounding $\mathrm{vol}(K_\delta)$. These objectives are achieved in Lemma 5.2 and Lemma 5.3. Lemma 5.4 bounds F_t above by a function of the $\mathrm{vol}(\partial K_\delta)$ and below by a function of $\mathrm{vol}(\partial K_{-\delta})$. Lemma 5.5 puts Lemmas 5.2, 5.3 and 5.4 together into one upper and one lower bound on F_t that involve more convenient quantities. In Lemma 5.6, we prove a bound on $\frac{V}{S}$ in terms of r_{in}. This together with Lemma 5.5 yields the desired proposition. The following fact is known (page 284, [21]) and will be useful in the proof of Lemma 5.2.

Lemma 5.1. *The surface area of a convex body is less than or equal to the surface area of any convex body that contains it.*

Lemma 5.2.

$$\mathrm{vol}(\partial K_{-\delta}) \geq \left(1 - n\frac{\delta}{r_{in}}\right) \mathrm{vol}(\partial K)$$

Proof. Let O be the center of the sphere of radius r_{in} contained inside K. We shall first prove that $K_{-\delta}$ contains $(1 - \frac{\delta}{r_{in}})K$ where this scaling is done from the origin O. Let A be a point on ∂K and let F be the image of A under this scaling. It suffices to prove that $F \in K_{-\delta}$.

We construct the smallest cone from A containing the sphere. Let B be a point where the cone touches the sphere. We have $OB = r_{in}$. Now consider the inscribed sphere centered at F. By similarity of triangles, we have

$$\frac{CF}{OB} = \frac{AF}{AO}.$$

Noticing that $AF = \frac{\delta}{r_{in}} OA$, we obtain

$$CF = OB\frac{AF}{AO} = \delta.$$

We thus see that the radius of the inscribed ball is δ and hence the δ-ball centered in F is contained in K. Therefore, $F \in K_{-\delta}$. Therefore by Lemma 5.1,

$$\mathrm{vol}(\partial K_{-\delta}) \geq \mathrm{vol}\left((1 - \frac{\delta}{r_{in}})\,\partial K\right). \tag{7}$$

Since the volumes of $n-1$-dimensional objects scale as $n-1^{th}$ powers and observing that for $x < 1$, $\max\{0, (1-x)^{n-1}\} > 1 - nx$, we see that

$$\mathrm{vol}\left((1 - \frac{\delta}{r_{in}})\,\partial K\right) = (1 - \frac{\delta}{r_{in}})^{n-1}\,\mathrm{vol}(\partial K) \geq (1 - \frac{n\delta}{r_{in}})\mathrm{vol}(\partial K).$$

□

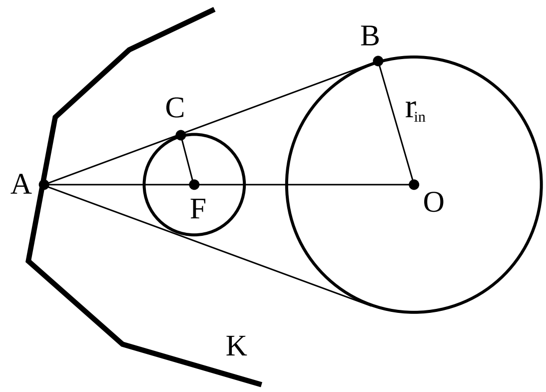

Figure 3: $K_{-\delta}$ contains $\left(1 - \frac{\delta}{r_{in}}\right) K$

Lemma 5.3. *The following bounds on the volume and surface area of the outer parallel bodies K_δ hold.*

$$\mathrm{vol}(K_\delta) \leq V \exp\left(\delta \frac{S}{V}\right).$$

$$\mathrm{vol}(\partial K_\delta) \leq S \exp\left(\delta \frac{S}{V}\right).$$

Proof. The volume and surface area of K_δ are polynomials in δ, given by the *Steiner formula* (see page 197, [21].)

$$\mathrm{vol}(K_\delta) = \sum_{i=0}^{n} \binom{n}{i} a_i \delta^i, \tag{8}$$

and

$$\mathrm{vol}(\partial K_\delta) = \sum_{i=1}^{n} i \binom{n}{i} a_i \delta^{i-1}. \tag{9}$$

From the Alexandrov-Fenchel inequalities (see Subsection 1.1) the coefficients a_i are log-concave; i. e.

$$a_i^2 \geq a_{i-1} a_{i+1}.$$

As a result

$$\frac{a_i}{a_0} \leq \left(\frac{a_1}{a_0}\right)^i.$$

Note that a_0 is V, the volume of K while na_1 is the surface area S of K. We thus have

$$\mathrm{vol}(K_\delta) \leq a_0 \left(\sum_{i=0}^{n} \binom{n}{i} \left(\frac{a_1 \delta}{a_0}\right)^i\right) \leq V e^{\frac{\delta S}{V}}, \tag{10}$$

and

$$\mathrm{vol}(\partial K_\delta) \leq a_1 \left(\sum_{i=1}^{n} i \binom{n}{i} \left(\frac{a_1 \delta}{a_0}\right)^{i-1}\right) \leq S e^{\frac{\delta S}{V}}. \tag{11}$$

□

Lemma 5.4.

$$\sqrt{\frac{\pi}{t}}\int_{\delta\geq 0}\mathrm{vol}(\partial K_{-\delta})e(t,\delta)d\delta < F_t < \sqrt{\frac{\pi}{t}}\int_{\delta\geq 0}\mathrm{vol}(\partial K_{\delta})e(t,\delta)d\delta. \tag{12}$$

Proof. We will denote the points in $\mathbb{R}^n$ by x, y, etc, and $n-$dimensional volume elements in integrals by $d\mathrm{vol}_x$, $d\mathrm{vol}_y$, etc. We have

$$\begin{aligned} F_t &= \sqrt{\frac{\pi}{t}}\int_K\int_{\mathbb{R}^n\setminus K} G_t(x,y)d\mathrm{vol}_y d\mathrm{vol}_x & (13)\\ &> \sqrt{\frac{\pi}{t}}\int_K e(t, dist(x,\partial K))d\mathrm{vol}_x, & (14)\end{aligned}$$

because for a fixed $x \in \partial K_{-\delta}$,

$$\int_{\mathbb{R}^n\setminus K} G_t(x,y)d\mathrm{vol}_y > \int_{H_x} G_t(x,y)d\mathrm{vol}_y = e(t,\delta).$$

The coarea formula [7] states the following.

Let Ω be a domain in $\mathbb{R}^n$. Let $g \in L^1$ and u be a real-valued Lipschitz function on Ω. Then,

$$\int_\Omega g(x)|\nabla(u)(x)|d\mathrm{vol}_x = \int_{-\infty}^{\infty}\left(\int_{u^{-1}(t)} g(x)dH^{n-1}(x)\right)dt. \tag{15}$$

Here H_{n-1} is the $n-1$ dimensional Hausdorff measure.

In particular, if g is constant on $u^{-1}(t)$ and (abusing notation) takes value $g(u^{-1}(t))$, this simplifies to

$$\int_\Omega g(x)|\nabla(u)(x)|d\mathrm{vol}_x = \int_{-\infty}^{\infty} g(u^{-1}(t))H^{n-1}(u^{-1}(t))dt. \tag{16}$$

Let Ω be the interior of K and $u(x) = dist(x,\partial K)$, and $g(x) = e(t, dist(x,\partial K))$. Then u is $1-$Lipshitz because of the triangle inequality applied to Hausdorff distance: for any $x, y \in \mathbb{R}^n$, $|u(x)-u(y)| = |\mathrm{dist}(x,\partial K) - dist(y,\partial K)| < |x-y|$. Therefore u is differentiable almost everywhere by Rademacher's Theorem. Suppose that u is differentiable at a point $x \in \Omega$. Then, because u has a gradient at x, there is a unique point z in ∂K nearest to x. However for y in a small neighborhood of x, $u(y) \leq |y-z|$, and $u(x) = |x-z|$. Since the gradient of $f_z(y) := |y-z|$ for $y = x$ is the vector $(x-z)/|x-z|$, which has norm 1, this proves that

$$|\nabla(u)(x)| \geq 1,$$

which together with the $1-$Lipschitz condition implies that

$$|\nabla(u)(x)| = 1. \tag{17}$$

By (16) and (17), we have

$$\sqrt{\frac{\pi}{t}}\int_K e(t, dist(x,\partial K))d\mathrm{vol}_x = \sqrt{\frac{\pi}{t}}\int_0^{\infty}\mathrm{vol}(\partial K_{-\delta})e(t,\delta)d\delta. \tag{18}$$

By the same token for a fixed $y \in \partial K_\delta$

$$\int_K G_t(x,y)d\mathrm{vol}_x < \int_{H_y} G_t(x,y)d\mathrm{vol}_x = e(t,\delta)$$

The following equality is obtained from the coarea formula in a similar manner. This time $\Omega = \mathbb{R}^n \setminus K$ and $u(x) = dist(x, \partial K)$, and $g(x) = e(t, dist(x, \partial K))$.

$$\sqrt{\frac{\pi}{t}} \int_{\mathbb{R}^n \setminus K} e(t, dist(x, \partial K)) d\text{vol}_x \quad = \quad \sqrt{\frac{\pi}{t}} \int_0^\infty \text{vol}(\partial K_\delta) e(t, \delta) d\delta. \tag{19}$$

Proceeding as before, we have the upper bound

$$F_t < \sqrt{\frac{\pi}{t}} \int_{\delta \geq 0} \text{vol}(\partial K_\delta) e(t, \delta) d\delta.$$

□

In order to provide some intuition for Lemma 5.5 we first show the following.

Claim 5.1. *$\lim_{t\to 0} F_t$ equals the surface area S of K.*

Proof. By Lemma 5.1, Lemma 5.2 and Lemma 5.3, we see that

$$\max(|\text{vol}(K_\delta) - S|, |\text{vol}(K_{-\delta}) - S|) \leq \exp(C_K \delta) - 1 \tag{20}$$

for some constant C_K that depends only on K but not δ. By Lemma 5.4 and (20),

$$\sqrt{\frac{\pi}{t}} \int_{\delta \geq 0} (S - (\exp(C_K\delta) - 1)) e(t, \delta) d\delta < F_t < \sqrt{\frac{\pi}{t}} \int_{\delta \geq 0} (S + (\exp(C_K\delta) - 1)) e(t, \delta) d\delta. \tag{21}$$

Thus

$$\left| F_t - \sqrt{\frac{\pi}{t}} \int_{\delta \geq 0} S e(t, \delta) d\delta \right| < \sqrt{\frac{\pi}{t}} \int_{\delta \geq 0} (\exp(C_K\delta) - 1) e(t, \delta) d\delta. \tag{22}$$

By (6) and (22), we see that

$$\begin{aligned}
\left| F_t - S\sqrt{\frac{\pi}{t}} \int_{\delta \geq 0} \delta N(t, \delta) d\delta \right| \quad &< \quad \sqrt{\frac{\pi}{t}} \int_0^\infty \left(\int_0^\delta (\exp(C_K\delta') - 1) d\delta' \right) N(t, \delta) d\delta & (23)\\
&= \quad \sqrt{\pi} \int_0^\infty \left(\int_0^\delta (\exp(C_K\delta') - 1) d\delta' \right) N(1, \delta/\sqrt{t}) d\delta & (24)\\
&= \quad \sqrt{\pi} \int_0^\infty \left(C_K^{-1} \exp(C_K\delta) - \delta - C_K^{-1} \right) N(1, \delta/\sqrt{t}) d\delta. & (25)
\end{aligned}$$

Since $\int_{\delta \geq 0} \delta N(t, \delta) = \sqrt{\frac{t}{\pi}}$, and as $t \to 0$, for any fixed $\epsilon > 0$

$$\left(C_K^{-1} \exp(C_K\delta) - \delta - C_K^{-1} \right) N(1, \delta/\sqrt{t}) = \left(C_K^{-1} \exp(C_K\delta) - \delta - C_K^{-1} \right) \frac{e^{-\frac{\delta^2}{4t}}}{\sqrt{4\pi}} \tag{26}$$

tends to 0 uniformly for $\delta \in [\epsilon, \infty)$ and is uniformly bounded for $\delta \in [0, \infty)$, we obtain

$$\lim_{t \to 0} |F_t - S| = 0. \tag{27}$$

□

Lemma 5.5. *Let $\alpha = \left(\frac{S}{V}\right)^2 t$. We then have*

$$S\left(1 - \frac{n\sqrt{\pi t}}{2r_{in}}\right) < F_t < S\left(\sqrt{\frac{\pi}{\alpha}} \frac{\exp(\alpha) - 1}{2} + \exp(\alpha)\right).$$

Proof. Applying (5) when $f(\delta) = \text{vol}(\partial K_\delta)$, we see that

$$\sqrt{\frac{\pi}{t}} \int_{\delta \geq 0} \text{vol}(\partial K_\delta) e(t, \delta) d\delta = \sqrt{\frac{\pi}{t}} \int_{\delta \geq 0} (\text{vol}(K_\delta) - \text{vol}(K)) N(t, \delta) d\delta. \tag{28}$$

Note that

$$\sqrt{\frac{\pi}{t}} \int_{\delta \geq 0} N(t, \delta) d\delta = \sqrt{\frac{\pi}{4t}}. \tag{29}$$

An application of Lemma 5.3 gives

$$F_t < \sqrt{\frac{\pi}{t}} \int_{\delta \geq 0} V \left(\exp\left(\delta \frac{S}{V}\right) - 1 \right) N(t, \delta) d\delta. \tag{30}$$

By (29) and (30), we have

$$\begin{aligned} F_t + V\sqrt{\frac{\pi}{4t}} < \sqrt{\frac{\pi}{t}} \int_{\delta \geq 0} V \exp\left(\frac{\delta S}{V}\right) N(t, \delta) d\delta &= \sqrt{\frac{\pi}{t}} \int_{\delta \geq 0} V \exp\left(\delta \sqrt{\frac{\alpha}{t}}\right) N(t, \delta) d\delta, & (31) \\ &= \sqrt{\frac{\pi}{t}} \int_{\delta \geq 0} V \left(\frac{\exp\left(\delta\sqrt{\frac{\alpha}{t}} - \frac{\delta^2}{4t}\right)}{\sqrt{4\pi t}} \right) d\delta. & (32) \end{aligned}$$

We split (32) into two parts, equating it to

$$\sqrt{\frac{\pi}{t}} \exp(\alpha) V \left[\int_{\delta=0}^{2\sqrt{\alpha t}} \left(\frac{\exp\left(-\left(\frac{\delta - 2\sqrt{\alpha t}}{2\sqrt{t}}\right)^2\right)}{\sqrt{4\pi t}} \right) d\delta + \int_{\delta = 2\sqrt{\alpha t}}^{\infty} \left(\frac{\exp\left(-\left(\frac{\delta - 2\sqrt{\alpha t}}{2\sqrt{t}}\right)^2\right)}{\sqrt{4\pi t}} \right) d\delta \right]. \tag{33}$$

By (32) and (33), we see that

$$F_t + V\sqrt{\frac{\pi}{4t}} < \sqrt{\frac{\pi}{t}} \exp(\alpha) V \left[\int_{\delta = -2\sqrt{\alpha t}}^{0} N(t, \delta) d\delta + \int_0^\infty N(t, \delta) d\delta \right]. \tag{34}$$

For $\delta \in (-\infty, 0]$ and any fixed t, $N(t, \delta)$ monotonically increases. Therefore,

$$\int_{\delta = -2\sqrt{\alpha t}}^{0} N(t, \delta) d\delta \leq N(t, 0)(2\sqrt{\alpha t}) = \sqrt{\frac{\alpha}{\pi}}. \tag{35}$$

We also know that

$$\int_0^\infty N(t, \delta) d\delta = \frac{1}{2}. \tag{36}$$

Note that

$$V = S\sqrt{\frac{t}{\alpha}}.$$

By (34) and (35), we now see that

$$\begin{aligned} F_t &< S e^\alpha + \sqrt{\frac{\pi}{4t}} V (e^\alpha - 1) & (37) \\ &= S \left(e^\alpha + \sqrt{\frac{\pi}{\alpha}} \left(\frac{e^\alpha - 1}{2} \right) \right), & (38) \end{aligned}$$

which proves the upper bound in the statement of the lemma.
Next, observe that Lemmas 5.2 and 5.4 imply that

$$F_t > \sqrt{\frac{\pi}{t}} \int_{\delta \geq 0} S\left(1 - n\frac{\delta}{r_{in}}\right) e(t,\delta) d\delta.$$

We transform this integral by parts using (6) and obtain

$$F_t > \sqrt{\frac{\pi}{t}} \int_{\delta \geq 0} S\left(\delta - n\frac{\delta^2}{2r_{in}}\right) N(t,\delta) d\delta = S\left(1 - \frac{n\sqrt{\pi t}}{2r_{in}}\right).$$

Applying a change of variables $u := \delta^2$, it is easy to verify that

$$\int_{\delta \geq 0} \delta N(t,\delta) d\delta = \sqrt{\frac{t}{\pi}}.$$

Also,

$$\int_{\delta \geq 0} \delta^2 N(t,\delta) d\delta = t,$$

since this is half of the the variance. Next, observe that Lemmas 5.2 and 5.4 imply that

$$F_t > \sqrt{\frac{\pi}{t}} \int_{\delta \geq 0} S\left(1 - n\frac{\delta}{r_{in}}\right) e(t,\delta) d\delta.$$

We transform this integral by parts using (6) and obtain the desired lower bound on F_t,

$$F_t > \sqrt{\frac{\pi}{t}} \int_{\delta \geq 0} S\left(\delta - n\frac{\delta^2}{2r_{in}}\right) N(t,\delta) d\delta = S\left(1 - \frac{n\sqrt{\pi t}}{2r_{in}}\right).$$

□

For proving Lemma 5.6 we will need the following claim.

Claim 5.2. *For any unit vector* u *that minimizes* $\frac{\|T^{-1}u\|}{\sqrt{2}}$, *if* x *is chosen uniformly at random from* K, $var(\mathrm{u} \cdot \mathrm{x}) \leq 5\mathrm{r}_{\mathrm{in}}^2$.

Proof. Since TK is $2-$isotropic, if x is chosen uniformly at random from TK, $\mathrm{var}(\mathrm{u} \cdot \mathrm{x}) \leq 2$. By definition r_{opt} is the smallest eigenvalue of $T^{-1}/\sqrt{2}$. Therefore, if x is chosen uniformly at random from TK,

$$\mathrm{var}\left(\frac{T^{-1}(\mathrm{u}) \cdot \mathrm{x}}{\sqrt{2}}\right) \leq 2r_{opt}^2.$$

Consequently, if x is chosen uniformly at random from K, then

$$\mathrm{var}(\mathrm{u} \cdot \mathrm{x}) \leq 4\mathrm{r}_{\mathrm{opt}}^2.$$

Since $r_{in} \geq \frac{2r_{opt}}{\sqrt{5}}$, we have $5r_{in}^2 \geq 4r_{opt}^2$. The preceding two sentences imply that

$$\mathrm{var}(\mathrm{u} \cdot \mathrm{x}) \leq 5\mathrm{r}_{\mathrm{in}}^2,$$

and the claim follows.

□

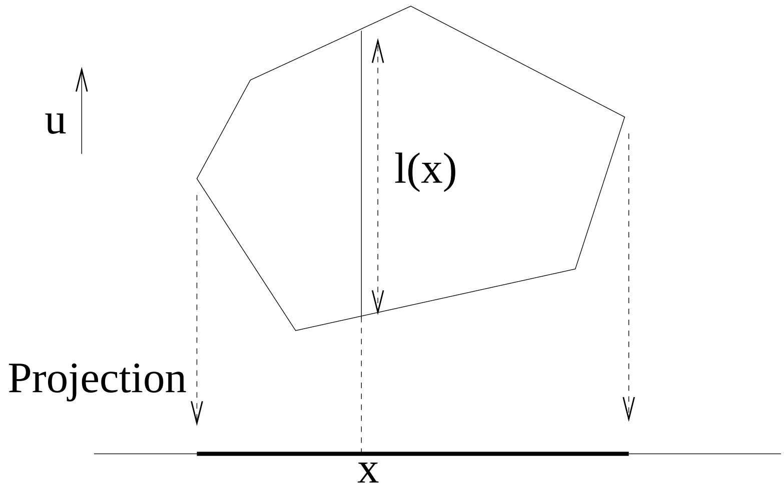

Figure 4: Projecting along a unit vector u minimizing $\|T^{-1}u\|$

Lemma 5.6.

$$\frac{r_{in}}{n} \leq \frac{V}{S} < 4\, r_{in}$$

Proof. It follows from Lemma 3.4 in [15] that a ball of radius $\frac{1}{\sqrt{2}}$ around the center of mass of TK is entirely contained in TK. Therefore

Observation 5.1. *K contains a ball of radius r_{in}.*

We are now in a position to present the proof of Lemma 5.6. The first inequality $\frac{r_{in}}{n} \leq \frac{V}{S}$ is the statement of Lemma 2.1. The only condition on r_{in} there, is that $r_{in}B \subseteq K$, a property that is satisfied by r_{in} by Observation 5.1.

Fix a unit vector u such that for x chosen uniformly at random from K, $\mathrm{var}(\mathrm{u}\cdot\mathrm{x}) \leq 5\mathrm{r}_{\mathrm{in}}^2$. By Claim 5.2, such a vector exists.

Let π be an orthogonal projection of K onto a hyperplane perpendicular to u. Further, for a point $y \in \pi(K)$, let ℓ_y be the length of the preimage $\pi^{-1}(y)$.

The variance of $\mathrm{u}\cdot\mathrm{x}$ under the condition $\pi(x) = y$, is given by $\ell_y^2/12$, since this is the variance of a random variable that takes a value from an interval of length ℓ_y uniformly at random.

$$\mathrm{var}(\mathrm{u}\cdot\mathrm{x}) = \frac{\int_{\pi(K)} \mathrm{var}(\mathrm{u}\cdot\mathrm{x}|\pi(\mathrm{x}) = \mathrm{y})\ell_{\mathrm{y}}\mathrm{dvol}_{\mathrm{y}}}{V} \tag{39}$$

$$= \frac{\int_{\pi(K)} \ell_y^3 d\mathrm{vol}_y}{12V}. \tag{40}$$

$$\tag{41}$$

$$\frac{\int_{\pi(K)} \ell_y^3 d\mathrm{vol}_y}{\mathrm{vol}(\pi(K))} \geq \left(\frac{\int_{\pi(K)} \ell_y d\mathrm{vol}_y}{\mathrm{vol}(\pi(K))}\right)^3 = \left(\frac{V}{\mathrm{vol}(\pi(K))}\right)^3,$$

since for any non-negative random variable X, $E[X^3] \geq E[X]^3$. Therefore,

$$5r_{in}^2 \geq \mathrm{var}(\mathrm{u}\cdot\mathrm{x}) = \frac{\int_{\pi(\mathrm{K})} \ell_{\mathrm{y}}^3 \mathrm{dvol}_{\mathrm{y}}}{12\mathrm{V}} \geq \left(\frac{\mathrm{V}^2}{12\mathrm{vol}(\pi(\mathrm{K}))^2}\right).$$

Further, $\text{vol}(\pi(K)) \leq S/2$. Hence,

$$5r_{in}^2 \geq \left(\frac{V^2}{12\text{vol}(\pi(K))^2}\right) \geq \left(\frac{V^2}{3S^2}\right),$$

and so $\frac{V}{S} < \sqrt{15}\, r_{in} < 4\, r_{in}$.

□

The lower bound on F_t is immediate for $\sqrt{t} = \frac{\epsilon r_{in}}{4n}$ using the lower bound in Lemma 5.5. To prove the upper bound, we observe that $\alpha = \left(\frac{S}{V}\right)^2 t \leq \left(\frac{n}{r_{in}}\right)^2 t$ from Lemma 5.4, which equals $\frac{\epsilon'^2}{16}$. Since $\epsilon < 0.5, \alpha < 1$. Therefore $\mathrm{e}^{\alpha} < 1 + 2\alpha$. It follows that

$$\begin{aligned} S\left(\sqrt{\frac{\pi}{\alpha}}\frac{\exp(\alpha)-1}{2} + \exp(\alpha)\right) &< S\left(\sqrt{\pi\alpha} + 1 + 2\alpha\right) \\ &< S(1 + 4\sqrt{\alpha}) \\ &< (1+\epsilon)S, \end{aligned}$$

thus proving Proposition 5.1. □

Proposition 5.2 states that the empirical quantity $\hat{S}$ computed by the surface area algorithm is likely to be an ϵ-approximation of F_t with probability $> 3/4$. Let $x_1, x_2, \dots, x_N$ from K, be ϵ'-uniform and each pair $\{x_i, x_j\}$ for $1 \leq i < j \leq N$ be μ-independent with probability $> 15/16$. Let $v_1, \dots, v_N$ be N independent random samples from the spherically symmetric multivariate Gaussian distribution whose mean is $\vec{0}$ and variance is $2nt$. Let $\hat{N} := |\{i | x_i + v_i \notin K\}|$. Then,

Proposition 5.2. *Let $\sqrt{t} = \frac{\epsilon' r_{in}}{4n}$ and $\epsilon' < 1/2$. Then, with probability greater than $\frac{3}{4}$,*

$$(1-\epsilon')(1-2\epsilon')F_t < \frac{\hat{N}}{N}\sqrt{\frac{\pi}{t}}\hat{V} < (1+\epsilon')(1+2\epsilon')F_t.$$

Proof. The proposition follows from the following lemmas.

The following lemma is a consequence of (Lemma 7.1, [23]) and summarizes some properties of μ-independence that we shall need.

Lemma 5.7.

1. *Let X and Y be μ-independent, and let f, g be two measurable functions. Then $f(X)$ and $g(Y)$ are also μ-independent.*

2. *Let X, Y be μ-independent random variables such that $0 \leq X \leq a$ and $0 \leq Y \leq b$. Then,*

$$|E(XY) - E(X)E(Y)| \leq \mu ab.$$

3. *Let $x_1, \dots, x_N$ be a Markov chain and assume that $\forall i > 0$, x_{i+1} is μ-independent from x_i. Then, $\forall i \neq j, x_i$ and x_j are μ-independent.*

Lemma 5.8. *Let $\sqrt{t} := \frac{\epsilon' r_{in}}{4n}$, where $\epsilon' < 1/2$. Then, with probability greater than 7/8,*

$$(1-\epsilon')F_t < E\left[\frac{\hat{N}}{N}\sqrt{\frac{\pi}{t}}V\right] < (1+\epsilon')F_t,$$

and $(1-\epsilon')V < \hat{V} < (1+\epsilon')V$.

Proof. Let X_i denote the indicator random variable for the event $x_i + v_i \notin K$. Suppose that x_i is sampled from the probability distribution p. x_i is $\frac{\epsilon'^2}{64n}$-uniform, therefore it has a distribution p for which $\int_K |p(x) - 1/V| d\text{vol}_x < \frac{\epsilon'^2}{64n}$. Therefore

$$\begin{aligned}
|\frac{\sqrt{t}F_t}{\sqrt{\pi}V} - E[X_i]| &= |\int_{\mathbb{R}^n \setminus K} \int_K G_t(x,y)(p(x) - \frac{1}{V}) d\text{vol}_x d\text{vol}_y| \\
&= |\int_K (p(x) - \frac{1}{V}) \int_{\mathbb{R}^n \setminus K} G_t(x,y) d\text{vol}_y d\text{vol}_x| \\
&< \int_K |(p(x) - \frac{1}{V})| d\text{vol}_x \\
&< \frac{\epsilon'^2}{64n}.
\end{aligned}$$

In the above calculations, the inequality is a consequence of the fact that

$$\int_{\mathbb{R}^n \setminus K} G_t(x,y) d\text{vol}_y < 1.$$

The calculations above hold for each i, and the lemma now follows from the linearity of expectation. □

Lemma 5.9. *With probability greater than* 15/16*,*

$$var\left(\frac{\hat{N}}{N}\sqrt{\frac{\pi}{t}}V\right) < \frac{\epsilon'^2 F_t^2}{16}.$$

Proof. Let X_i denote the indicator random variable for the event $x_i + v_i \notin K$ as in the proof of Lemma 5.8. Then, Let

$$q = \frac{F_t}{V}\sqrt{\frac{t}{\pi}}$$

and $\epsilon'' = \epsilon' q$.

$$\text{var}(\frac{\sum_1^N X_i}{N}) = \frac{\sum_1^N \text{var} X_i + \sum_{i \neq j} \text{cov}(X_i, X_j)}{N^2}.$$

Since we are dealing with $0, 1$ variables,

$$\text{var}(X_i) < E[X_i] < q + \frac{\epsilon'^2}{64n} < q(1 + \epsilon').$$

The last statement follows from Lemma 5.8, and the fact that $\frac{\epsilon'}{64n} < q$, which we show below.

$$\frac{F_t}{V}\sqrt{\frac{t}{\pi}} > \frac{S\sqrt{t}}{2\sqrt{\pi}V} > \frac{S\sqrt{t}}{4V},$$

as a consequence of Proposition 5.1. Using the values of $\sqrt{t}$ and the inequality $\frac{V}{S} < 4r_{in}$ from Lemma 5.6, this is $> \frac{\epsilon'}{64n}$.

Continuing the proof, from Lemma 5.7 we have that,

$$\frac{\sum_{i \neq j} \text{cov}(X_i, X_j)}{N^2} < \mu.$$

$$\frac{\sum_1^N(\frac{F_t}{V}\sqrt{\frac{t}{\pi}}+\epsilon'')+N(N-1)\mu}{N^2}<$$

$$\frac{\frac{F_t}{V}\sqrt{\frac{t}{\pi}}}{N}+\frac{\epsilon''}{N}+\mu<\frac{2q}{N}+\mu.$$

We just showed that $q=\frac{\epsilon''}{\epsilon'}>\frac{\epsilon'}{64n}$. We are required to show that

$$\frac{2q}{N}+\mu<\frac{\epsilon'^2q^2}{16},$$

The ratio of the left hand side to the right,

$$\frac{32}{N\epsilon'^2q}+\frac{16\mu}{\epsilon'^2q^2}<\frac{2^{11}n}{N\epsilon'^3}+\frac{2^{16}\mu n^2}{\epsilon'^4}=1/2<1.$$

□

Using Chebycheff's inequality and Lemma 5.9,

$$P\left[\left|\frac{\hat{N}}{N}\sqrt{\frac{\pi}{t}}V-E\left[\frac{\hat{N}}{N}\sqrt{\frac{\pi}{t}}V\right]\right|>\epsilon' F_t\right]<\frac{1}{16}.$$

This statement, together with Lemma 5.8 completes the proof of Proposition 5.2. □

We now proceed to prove our main theorem, Theorem 3.1, which we recall below.

Theorem 3.1. *The surface area of a convex body K, given by a membership oracle, and parameters r, R such that $rB\subseteq K\subseteq RB$ can be approximated to within a relative error of ϵ with probability $1-\delta$ using at most*

$$O\left(n^4\log\frac{1}{\delta}\left(\frac{1}{\epsilon^2}\log^9\frac{n}{\epsilon}+\log^8 n\log\frac{R}{r}+\frac{1}{\epsilon^3}\log^7\left(\frac{n}{\epsilon}\right)\right)\right)$$

i. e. $O^*(n^4)$ oracle calls.

Proof of Theorem 3.1. Propositions 5.1 and 5.2 together imply that with probability $>3/4$, the output

$$\hat{S}=\frac{\hat{N}}{N}\sqrt{\frac{\pi}{t}}\hat{V},$$

satisfies

$$(1-\epsilon)S<\hat{S}<(1+\epsilon)S.$$

Running the algorithm (whose confidence is 3/4) $\lceil 36\ln\left(\frac{2}{\delta}\right)\rceil$ times and taking the median of the outputs gives the result with a confidence $>1-\delta$. Let m independent trials be made of an event whose probability of success is p and $\hat{m}$ be the number of resulting successes. Then, it follows from Hoeffding's inequality (Theorem 1 in [10]) that

$$P\left[\left|\frac{\hat{m}}{m}-p\right|\geq\lambda p\right]\leq 2\exp\left(\frac{-\lambda^2mp}{3}\right). \quad (42)$$

We will consider repeated trials of the algorithm, and declare the event to be a success if the output $\hat{S}$ of the algorithm lies in $[(1-\epsilon)S,(1+\epsilon)S]$. In order for the median of all the trials to lie in $[(1-\epsilon)S, (1+\epsilon)S]$, it suffices for at least half of the outputs to lie here. Setting $m = \lceil 36 \ln\left(\frac{2}{\delta}\right) \rceil$, $p \geq \frac{3}{4}$ and $\lambda = 1/3$, gives us

$$2 \exp\left(\frac{-\lambda^2 m p}{3}\right) \leq \delta,$$

which translates into a probability of overall success that exceeds $1 - \delta$.

The bounds on the run-time (involving the computation of $\hat{N}$ and $\hat{V}$), follow from the analysis in Subsection 4.3. This completes the proof of Theorem 3.1. □

6 Concluding Remarks

We obtained a randomized algorithm for estimating the surface area of a convex body given by a membership oracle, with better complexity guarantees than the previous algorithm of [5]. The algorithm can be interpreted as a simulation of heat flow over a small time out of a hot body. These results can be partially extended to a setting involving hypersurfaces. The role of convexity is then played by bounded "reach", where the reach of the hypersurface is defined as the largest number t such that any point at a distance atmost t from the hypersurface has a unique nearest point on the hypersurface. Details may be found in [19] and [17]. Connections to graph cuts have been explored in [18] and [17], wherein the amount of heat diffusing across a surface has been related to the size of a graph cut.

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References

[1] I. Bárány and Z. Furedi, "Computing the Volume is Difficult." Discrete and Computational Geometry 2, 1987, 319-326.

[2] M. Belkin, H. Narayanan and P. Niyogi, "Heat Flow and a Faster Algorithm to Compute the Surface Area of a Convex Body", Proceedings of the 47th Annual IEEE Symposium on Foundations of Computer Science, 2006

[3] D. Bertsimas and S. Vempala, "Solving convex programs by random walks" Journal of the ACM (JACM) 51(4), 540–556, 2004. Proc. of the 34th ACM Symposium on the Theory of Computing (STOC '02), Montreal, 2002.

[4] M. Dyer, A. Frieze and R. Kannan, "A random polynomial time algorithm for approximating the volume of convex sets" (1991) in Journal of the Association for Computing Machinery, 38:1-17,

[5] M. Dyer, P. Gritzmann and A. Hufnagel (1998) "On the complexity of computing Mixed Volumes." Siam Journal of Computing., Vol. 27, No. 2, pp. 356-400, April 1998

[6] G. Elekes, "A Geometric Inequality and the Complexity of Computing Volume." Discrete and Computational Geometry 1, 1986, 289-292.

[7] H. Federer, "Curvature Measures," Transactions of the American Mathematical Society, Vol. 93, No. 3, 418–491.

[8] L. Gurvits, "A Polynomial-Time Algorithm to Approximate the Mixed Volume within a Simply Exponential Factor." Discrete and Computational Geometry, Volume 41, Number 4 (2009), 533-555,

[9] L. Gurvits and A. Samorodnitsky, "A Deterministic Algorithm for Approximating the Mixed Discriminant and Mixed Volume, and a Combinatorial Corollary." Discrete and Computational Geometry, 27(4): 531-550

[10] W. Hoeffding, "Probability Inequalities for Sums of Bounded Random Variables," Journal of the American Statistical Association, Vol. 58, No. 301 (Mar., 1963), pp. 13-30

[11] M. R. Jerrum, L. G. Valiant and V. V. Vazirani (1986), "Random generation of Combinatorial structures from a uniform distribution." Theoretical Computer Science, 43, 169-188

[12] R. M. Karp and M. Luby, (1983). "Monte-Carlo algorithms for enumeration and reliablility problems." Proc. of the 24th IEEE Foundations of Computer Science (FOCS '83),56-64

[13] M. Grötschel, L. Lovász, and A. Schrijver, Geometric algorithms and combinatorial optimization, Springer-Verlag, Berlin, 1988.

[14] L. Lovász and S. Vempala (2004), "Hit-and-run from a corner" Proc. of the 36th ACM Symposium on the Theory of Computing, Chicago

[15] L. Lovász and S. Vempala, "Simulated annealing in convex bodies and an $O^*(n^4)$ volume algorithm" Proc. of the 44th IEEE Foundations of Computer Science (FOCS '03), Boston, 2003.

[16] L. Lovász and S. Vempala, "Fast Algorithms for Logconcave Functions: Sampling, Rounding, Integration and Optimization", Proc. of the 47th IEEE Symposium on Foundations of Computer Science (FOCS '06), 2006.

[17] H. Narayanan, "Diffusion in Computer Science and Statistics," *Ph.D thesis*, University of Chicago, August 2009

[18] H. Narayanan, M. Belkin, P. Niyogi, "On the relation between Low Density Separation, Spectral Clustering and Graph Cuts," Proceedings of Neural Information Processing Systems, 2006

[19] H. Narayanan, and P. Niyogi, "Sampling hypersurfaces through diffusion," Proceedings of the 12th international workshop, on Approximation, Randomization and Combinatorial Optimization: Algorithms and Techniques, RANDOM'08, 2008

[20] V. Y. Pan, Z. Chen and A. Zheng, "The Complexity of the Algebraic Eigenproblem", MSRI Preprint 1998-71, Mathematical Sciences Research Institute, Berkeley, California (1998).

[21] R. Schneider, "Convex bodies: The Brunn-Minkowski Theory," Encyclopedia of Mathematics and its Applications, Cambridge University Press 1993.

[22] P. M. Vaidya, "A new algorithm for minimising convex functions over convex sets." Mathematical Programming, 73, 291-341, 1996.

[23] S. Vempala, "Geometric Random Walks: A Survey," Combinatorial and Computational Geometry, MSRI Publications, Volume 52, 2005.

[24] J. M. Wills, “Zum Verhaltnis von Volumen zur Oberfläche bei konvexen Körpern, Arch. Math. , 21, (1970), 557-560