

# Stat-491-Fall2014-Assignment-VI

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Note: This assignment consists of practice problems with solutions on the exponential distribution and the Poisson process. Please try the problems before looking at the solutions.

## 1 Preliminaries

### 1.1 Exponential distribution

1. We say  $T = \text{exp}(\lambda)$ , if  $P(T \leq t) = 1 - e^{-\lambda t}, \forall t \geq 0$ .
2. If  $T = \text{exp}(\lambda)$ , then its density function

$$f_T(t) = \lambda e^{-\lambda t}, t \geq 0; f_T(t) = 0, t < 0.$$

$$E[T] = 1/\lambda; E[T^2] = 2/(\lambda)^2; E[T^n] = n!/(\lambda)^n;$$

$$\text{Var}[T] = 1/(\lambda)^2.$$

3. (**memory less**)

$$P(T > t + s \mid T > t) = P(T > s)$$

4. ( $P(\min(S, T) > t), S, T$  independent)

Let  $S = \text{exp}(\mu), T = \text{exp}(\lambda)$ . Then, if they are also independent,

$$P(\min(S, T) > t) = P(S > t, T > t) = P(S > t)P(T > t) = e^{-\mu t}e^{-\lambda t} = e^{-(\lambda+\mu)t}.$$

More generally given independent random variables  $T_i = \text{exp}(\lambda_i)$ , the probability that the minimum of them is greater than  $t$  is  $e^{-\sum_i \lambda_i t}$ .

5. ( $P(S < T), S, T$  independent)

$$= \lambda/(\lambda + \mu).$$

More generally, the probability that among the independent random variables  $T_i = \text{exp}(\lambda_i), T_j$  is the minimum, is

$$\lambda_j / \sum_i \lambda_i.$$

6. If  $\tau_1, \tau_2, \dots$  are independent  $\exp(\lambda)$  random variables, then the sum  $T_n = \tau_1 + \tau_2 + \dots + \tau_n$  has a  $\text{gamma}(n, \lambda)$  distribution, its density function being

$$f_{T_n}(t) = \lambda e^{-\lambda t} [(\lambda t)^{n-1} / (n-1)!] \text{ for } t \geq 0 \text{ and } 0 \text{ for } t < 0.$$

## 2 Problems

### 2.1 Problems: Exponential distribution

Note: These problems are from Durrett's book.

- The time to repair a machine is exponentially distributed random variable with mean 2.
  - What is the probability the repair takes more than  $2h$ .
  - What is the probability that the repair takes more than  $5h$  given that it takes more than  $3h$ .
- The lifetime of a radio is exponentially distributed with mean 5 years. If Ted buys a 7 year-old radio, what is the probability it will be working 3 years later?
- A doctor has appointments at 9 and 9 : 30. The amount of time each appointment lasts is exponential with mean 30 *min*. What is the expected amount of time after 9 : 30 until the second patient has completed his appointment?
- Copy machine 1 is in use now. Machine 2 will be turned on at time  $t$ . Suppose that the machines fail at rate  $i$ . What is the probability that machine 2 is the first to fail?
- (a) Let  $S$  and  $T$  be independent and exponentially distributed with rates  $\lambda$  and  $\mu$ . Find the probability distribution for  $V = \max(S, T)$  and  $E[V]$ .  
Suppose  $S_1, \dots, S_n$  are independent exponentially distributed with rate  $\lambda$ . (b) Find the probability distribution for  $V = \max(\min(S_1, \dots, S_{n-1}), S_n)$  and  $E[V]$ .

### Solutions

- Let  $T$  be the time of completion of repair. we are given  $P(T \leq t) = 1 - e^{-t/2}$ . So  $P(T \geq t) = e^{-t/2}$ . (Here  $t$  is taken to be in hrs.) So
  - probability repair takes more than  $2hrs$  equals  $e^{-2/2} = e^{-1}$ ,
  - probability repair takes more than  $5hrs$  given that it takes more than  $3hrs$ , is  $P(T > 5, T > 3) / P(T > 3) = e^{-5/2} / e^{-3/2} = e^{-1}$ .
- $P[T > 10 \mid T > 7] = P[T > 3] = e^{-3/5}$ .
- Let  $T_2$  denote the sum of the two exponential distributions  $\tau_1, \tau_2$ , with mean  $1/\lambda$ .  $T_2$  has the probability distribution which has the density function  $f_2(t) = \lambda^2 t e^{-\lambda t}$ . By integrating we get,

$$P(T_2 \leq t) = 1 - e^{-\lambda t} - \lambda t e^{-\lambda t}.$$

So

$$P(T_2 > t) = e^{-\lambda t} + \lambda t e^{-\lambda t}.$$

The expected value of  $T_2$  is obtained by integrating the expression for  $P(T_2 > t)$ . This yields  $2/\lambda = 60$ . This expected value =  $E[T_2 | T_2 < 30] \times P(T_2 < 30) + E[T_2 | T_2 > 30] \times P(T_2 > 30)$ . However if  $E[T_2 | T_2 > 30]$  is asked for, then the answer is  $\int_{30}^{\infty} P(T_2 > t)dt/P(T_2 > 30)$ .

4. Probability that copy machine 2 fails first

$$\begin{aligned} &= Pr\{\text{machine 1 does not fail before } t, \text{ machine 2 is the first to fail after } t\} \\ &= P(2 \text{ fails first after } t | 1 \text{ does not fail before } T) \times P(1 \text{ does not fail before } t). \end{aligned}$$

After time  $t$ , the machines follow the distribution  $P(T_i \leq t + \tau) = 1 - e^{-\lambda_i \tau}$ ,  $i = 1, 2$ .

The probability that, after time  $t$ ,  $T_2 < T_1$  is  $\int_0^{\infty} f_{T_2}(\tau)P(T_1 > \tau)d\tau = \lambda_2/(\lambda_1 + \lambda_2)$ .

The probability that 1 does not fail before  $t$  is  $e^{-\lambda_1 t}$ .

So Probability that copy machine 2 fails first =  $\lambda_2/(\lambda_1 + \lambda_2) \times e^{-\lambda_1 t}$ .

5.  $P(V < t) = P(S < t, T < t) = (1 - e^{-\lambda t})(1 - e^{-\mu t}) = e^{-(\lambda+\mu)t} + 1 - e^{-\lambda t} - e^{-\mu t}$ .

$$\begin{aligned} E(V) &= \int_0^{\infty} P(V > t)dt = \int_0^{\infty} [-e^{-(\lambda+\mu)t} + e^{-\lambda t} + e^{-\mu t}]dt \\ &= 1/\lambda_1 + 1/\lambda_2 - 1/(\lambda_1 + \lambda_2). \end{aligned}$$

We could have done this also by  $V + U = S + T$ , where  $U = \min\{S, T\}$ .

So  $E(V) = E(S) + E(T) - E(U) = 1/\lambda_1 + 1/\lambda_2 - 1/(\lambda_1 + \lambda_2)$ .

(Note that linearity of expectation does not require independence of the random variables involved.)

(b)  $V = \min(S_1, \dots, S_{n-1})$  has rate  $\mu = \lambda_1 + \dots + \lambda_{n-1}$ .

$W = \max(V, S_n)$  has cumulative distribution  $(1 - e^{-\mu t})(1 - e^{-\lambda_n t})$ .

$E[W] = 1/\mu + 1/\lambda - 1/(\mu + \lambda)$ .

## 2.2 Problems:Poisson distribution

Below we have listed out standard properties of Poisson distribution. It is suggested that you try to prove them yourself before you look up any standard text book.

Prove

1. If  $X$  has probability distribution  $Poisson(\mu)$ , then

$$\mu = E[X] = Var[X].$$

2. If  $X, Y$  are independent and have probability distributions  $Poisson(\mu), Poisson(\lambda)$ , then  $X + Y$  has probability distribution  $Poisson(\mu + \lambda)$ .

3. If  $X, Y$  are independent and have probability distributions  $Poisson(\mu), Poisson(\lambda)$ , then given  $X + Y = k, X \sim Binom(k, \mu/(\lambda + \mu))$ , i.e.,

$$P(X = r) = \binom{k}{r} (\mu/(\lambda + \mu))^r (1 - \mu/(\lambda + \mu))^{k-r}, r \leq k.$$

4. Let  $\lambda = np$ . Then

$$\lim_{n \rightarrow \infty} \binom{n}{k} (\lambda/n)^k (1 - \lambda/n)^{n-k} = e^{-\lambda} (\lambda^k)/k!.$$

So for large  $n$  and small  $p$  and small  $k$ , the binomial distribution can be approximated by Poisson distribution, i.e.,  $Binom(n, \lambda/n)$  is close to  $Poisson(\lambda)$ .

### 2.3 Problems: Poisson process

- Suppose  $N(t)$  is a Poisson process with rate 3. Let  $T_n$  denote the time of the  $n^{th}$  arrival. Find
  - $E[T_{12}]$ ,
  - $E[T_{12} | N(2) = 5]$ ,
  - $E[N(5) | N(2) = 5]$ .
- Ellen catches fish at times of a Poisson process with rate 2 per hour. Forty percent of the fish are salmon, while sixty percent of the fish are trout. What is the probability she will catch exactly one salmon and two trout if she fishes for 2.5h?
- Signals are transmitted according to a Poisson process with rate  $\lambda$ . Each signal is successfully transmitted with probability  $p$  and lost with probability  $1 - p$ . Different signals may be regarded as independent. For  $t \geq 0$  let  $N(t)$  be the number of signals successfully transmitted and let  $N_2(t)$  be the number that are lost upto time  $t$ .  
Find the distribution of  $(N_1(t), N_2(t))$ . (b) What is the distribution of  $L =$  the number of signals lost before the first signal is successfully transmitted?
- Let  $Y_1, Y_2, \dots$  be independent and identically distributed, let  $N$  be an independent nonnegative integer valued random variable, and let  $S = Y_1 + \dots + Y_N$  with  $S = 0$  when  $N = 0$ .
  - If  $E[|Y_i|], E[N] < \infty$ , then  $E[S] = E[N] \cdot E[Y_i]$ .
  - If  $E[Y_i^2], E[N^2] < \infty$ , then  $var(S) = E[N]var(Y_i) + var(N)(E[Y_i])^2$ .
  - If  $N$  is  $Poisson(\lambda)$ , then  $var(S) = \lambda E[Y_i^2]$ .

### Solutions

- $E[T_{12}] =$  Mean of the sum of 12 exponential distributions of rate 3. For each of them the  $\lambda$  value is 1/3. So  $E[T_{12}] = 12/3 = 4$ .
  - $E[T_{12} | N(2) = 5] =$  the expected time of arrival of 12 - 5 events after time 2.  $E[T_7] = 7/3, E[T_{12} | N(2) = 5] = 2 + 7/3$ .
  - $E[N(5) | N(2) = 5] =$  expected number of arrivals in 5 - 2 seconds + 5. Expected number of arrivals in 3 seconds =  $3 \times$  rate = 9. So  $E[N(5) | N(2) = 5] = 9 + 5 = 14$ .
- Original Poisson process of fishing is rate 2 per hour.  
Salmon fishing is a Poisson process with rate  $2 \times 0.4 = 0.8$  per hour.  
Trout fishing is a Poisson process with rate  $2 \times 0.6 = 1.2$  per hour.  
The time given is 2.5hr.  $P(N_S = 1) = e^{-0.8 \times 2.5} (0.8 \times 2.5)$ .  $P(N_T = 2) = e^{-1.2 \times 2.5} (1.2 \times 2.5)^2 / 2$ .  
So the probability that  $N(S) = 1, N(T) = 2$  is  $2e^{-2} + (9/2)e^{-3}$ .
- $P(N_1(t) = k, N_2(t) = r) = (e^{-p\lambda t} (p\lambda t)^k / k!) (e^{-(1-p)\lambda t} ((1-p)\lambda t)^r / r!)$ .

4. i. We have

$$\begin{aligned} E[S] &= \sum_{n=0}^{\infty} E[S | N = n] \times P(N = n) \\ &= \sum_{n=0}^{\infty} n E[Y_i] \times P(N = n) = E[N] E[Y_i]. \end{aligned}$$

ii.  $\text{var}[S] \equiv E[S^2] - (E[S])^2$ . If  $S = Y_1 + \dots + Y_n, Y_i$  i.i.d., then  $\text{var}[S] = n(\text{var}[Y_i])$ . Now,  $E[S^2 | N = n] = \text{var}[S | N = n] + (E[S | N = n])^2 = n \text{var}[Y_i] + (nE[Y_i])^2$ . Therefore

$$\begin{aligned} E[S^2] &= \sum_{n=0}^{\infty} E[S^2 | N = n] \times P(N = n) \\ &= \sum_{n=0}^{\infty} (n \text{var}[Y_i] + n^2 (E[Y_i])^2) \times P(N = n) \\ &= (E[N]) \text{var}[Y_i] + (E[Y_i])^2 \times E[N^2]. \end{aligned}$$

Therefore

$$\begin{aligned} \text{var}[S] &\equiv E[S^2] - (E[S])^2 = (E[N]) \text{var}[Y_i] + (E[Y_i])^2 \times E[N^2] - (E[N] E[Y_i])^2 \\ &= (E[N]) \text{var}[Y_i] + (E[Y_i])^2 \times (E[N^2] - (E[N])^2) \\ &= (E[N]) \text{var}[Y_i] + \text{var}[N] (E[Y_i])^2. \end{aligned}$$

iii. If  $N$  is *Poisson*( $\lambda$ ), then  $E[N] = \text{var}[N] = \lambda$ . So

$$\text{var}[S] = \lambda(\text{var}[Y_i] + (E[Y_i])^2) = \lambda(E[Y_i^2]).$$