Stat-491-Fall2014-Assignment-V

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Note: This assignment is due on 3 December 2014.

1 Martingales

In what follows take $\mathcal{F}_n$ to be the sequence of random variables $X_0, \cdots X_n$. As usual all random variables are taken to be integrable with respect to the relevant $\sigma-$ algebras.

1. Let M_n be a martingale with respect to $\mathcal{F}_n$. Prove that $M'_n \equiv M_{m+n} - M_m, n \geq 1$, is a martingale with respect to $\mathcal{F}'_n \equiv X_0, \cdots X_{m+n}$.

   Solution

   We need to prove that $E[M'_n \mid \mathcal{F}'_{n-1}] = M'_{n-1}$. Now $E[M_{m+n} \mid \mathcal{F}'_{n-1}] = M_{m+n-1}$. Further, $E[M_m \mid \mathcal{F}'_{n-1}] = M_m$. Therefore, $E[M'_n \mid \mathcal{F}'_{n-1}] = E[M_{m+n} - M_m \mid \mathcal{F}'_{n-1}] = M_{m+n-1} - M_m = M'_{n-1}$.

2. Let Z be a random variable and let $M_n \equiv E[Z \mid \mathcal{F}_n]$. Prove that M_n is a martingale with respect to $\mathcal{F}_n$.

   Solution

   We need to prove that $E[M_n \mid \mathcal{F}_{n-1}] = M_{n-1}$. We have,

   $$E[M_n \mid \mathcal{F}_{n-1}] = E[E[Z \mid \mathcal{F}_n] \mid \mathcal{F}_{n-1}]$$

   $= E[Z \mid \mathcal{F}_{n-1}]$ (by second form of law of total probability) $= M_{n-1}$.

3. Let $Y_1, \cdots, Y_n$, be iid random variables with mean μ and let $X_0, \cdots, X_n$ be random variables such that $X_{n+1} = \Sigma_1^{X_n} Y_i$. Prove

   (a) $E[X_{n+1} \mid X_n] = \mu X_n$.

   (b) $M_n \equiv \mu^{-n} X_n$ is a martingale relative to $\mathcal{F}_n$.

   Solution

   (a) We have

   $$E[X_{n+1} \mid X_n = k] = E[\Sigma_1^{X_n} Y_i \mid X_n = k] = E[\Sigma_1^{k} Y_i \mid X_n = k] = E[kY_i] = kE[Y_i] = k\mu.$$

   We know that $E[X_{n+1} \mid X_n]$ is defined to be that random variable which when $X_n = k$, takes value $E[X_{n+1} \mid X_n = k] = \mu k$. Now $\mu X_n = \mu k$, when $X_n = k$. So $E[X_{n+1} \mid X_n] = \mu X_n$.

(b) We have

$$E[M_{n+1} \mid \mathcal{F}_n] = E[\mu^{-(n+1)} X_{n+1} \mid X_n] = \mu^{-(n+1)} E[X_{n+1} \mid X_n] = \mu^{-(n+1)}[\mu X_n] = M_n.$$

4. Let $Y_1, Y_2, \cdots$ be a sequence of independent random variables with zero mean and common variance σ^2. If $X_n = Y_1 + \cdots + Y_n$, then show that $X_n^2 - n\sigma^2$ is a martingale.

Solution

Let $M_n \equiv X_n^2 - n\sigma^2$. We have

$$X_{n+1}^2 - (n+1)\sigma^2 = (Y_1 + \cdots Y_n + Y_{n+1})^2 - (n+1)\sigma^2$$

$$= (X_n + Y_{n+1})^2 - n\sigma^2 - \sigma^2 = M_n + 2X_n Y_{n+1} + Y_{n+1}^2 - \sigma^2.$$

So

$$E[M_{n+1} \mid \mathcal{F}_n] = E[M_{n+1} \mid X_n] = E[M_n + 2X_n Y_{n+1} + Y_{n+1}^2 - \sigma^2 \mid X_n]$$

$$= E[M_n \mid X_n] + E[2X_n Y_{n+1} + Y_{n+1}^2 - \sigma^2 \mid X_n].$$

Now X_n, Y_{n+1} are independent and further Y_{n+1} has zero mean and variance σ^2. So $E[2X_n Y_{n+1} \mid X_n] = 2E[X_n][Y_{n+1}] = 0$, and $E[2X_n Y_{n+1} + Y_{n+1}^2 - \sigma^2 \mid X_n] = 0$. Therefore, $E[M_{n+1} \mid \mathcal{F}_n] = E[M_n \mid X_n] = M_n$.

5. Let $X_0, \cdots X_n, \cdots$ be a sequence of independent random variables taking values $0, 1, 2$ with probabilities respectively, p_0, p_1, p_2. The process stops when a subsequence s' appears for the first time. Find the expected stopping time $E[T]$ in each of the following cases:

(a) $s' \equiv 1, 2, 3$.

(b) $s' \equiv 2, 2, 2$.

(c) $s' \equiv 1, 2, 1$.

(d) $s' \equiv 2, 1, 1$.

Solution

The betting scheme is as follows. Bet \$1 for the first element. If it comes out right put the payoff on the second element and if that also succeeds put the payoff on the third element. If the first element is wrong you lose your dollar. There are bettors $0, 1, 2, \cdots$ who come into action at the 0^{th} place, first place, second place etc respectively. The martingale M_n is the sum of the profit of all the bettors at time n. Further $M_0 = 0$. It can be shown that the OST is applicable in this case because $E[T]$ can be shown to be finite and $E[|M_{n+1} - M_n| \mid \mathcal{F}_n]$ is bounded.

At stopping time T the situation is as follows: All T bettors have lost 1, and some have gained.

(a) The $(T-2)^{th}$ bettor has had payoff, in the case of end sequence $1, 2, 3$, equal to $1/p(1) \times 1/p(2) \times 1/p(3)$. In this case $(T-1)^{th}$ and T^{th} bettors, since they bet on the first element 1 and the outcome is not equal to 1 do not get any payoff. Therefore $E[M_T] = 1/p(1) \times 1/p(2) \times 1/p(3) - E[T] = M_0 = 0$. So $E[T] = 1/p(1) \times 1/p(2) \times 1/p(3)$.

(b) The $(T-2)^{th}$ bettor has had payoff, in the case of end sequence $2, 2, 2$ equal to $1/p(2) \times 1/p(2) \times 1/p(2)$. In this case $(T-1)^{th}$ and T^{th} bettors, since they bet on the first element 2 and the pair $2, 2$ and the outcome is equal to 2, and $2, 2$ get payoffs $1/p(2), (1/p(2))^2$ respectively. So $0 = E[M_T] = E[T] - 1/p(2) - (1/p(2))^2 - (1/p(2))^3$. Therefore $E[T] = 1/p(2) + (1/p(2))^2 + (1/p(2))^3$.

(c) $E[T] = (1/p(1))^2 \times 1/p(2) + 1/p(1)$.

(d) $E[T] = (1/p(1))^2 \times 1/p(2)$.

6. How would you solve the previous problem using the Markov chain ideas discussed in class?

Solution

We could build a markov chain with states which are three element sequences- in this case sequences of the kind $(0, 0, 0), (0, 1, 0)$ etc. and in general (a, b, c) where a, b, c take values $0, 1, 2$. There will be 27 such states.

From the state (i, b, c), we can only go to a state of the kind (b, c, j), and the probability of transition would be $p((i, b, c), (b, c, j)) = p(j)$.

In this Markov chain we can make the end sequence an absorbing state by removing all outgoing edges and putting a self loop of value 1 on it. Now we could find the hitting time from every state to the end state by writing linear equations for $l(i, R)$. The average of these times , i.e., $1/(number\ of\ states - 1) \times \Sigma hitting\ times = 1/27 \times \Sigma hitting\ times$, is the answer.