Stat-491-Fall2014-Assignment-V

Hariharan Narayanan

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Note: This assignment is due on 3 December 2014.

1 Martingales

In what follows take $\mathcal{F}_n$ to be the sequence of random variables $X_0, \cdots X_n$. As usual all random variables are taken to be integrable with respect to the relevant $\sigma-$ algebras.

1. Let M_n be a martingale with respect to $\mathcal{F}_n$. Prove that $M'_n \equiv M_{m+n} - M_m, n \geq 1$, is a martingale with respect to $\mathcal{F}'_n \equiv X_0, \cdots X_{m+n-1}$.

2. Let Z be a random variable and let $M_n \equiv E[Z \mid \mathcal{F}_n]$. Prove that M_n, is a martingale with respect to $\mathcal{F}_n$.

3. Let $Y_1, \cdots, Y_n$, be iid random variables with mean μ and let $X_0, \cdots, X_n$ be random variables such that $X_{n+1} = \Sigma_1^{X_n} Y_i$. Prove

   (a) $E[X_{n+1} \mid X_n] = \mu X_n$.

   (b) $M_n \equiv \mu^{-n} X_n$ is a martingale relative to $\mathcal{F}_n$.

4. Let $Y_1, Y_2, \cdots$ be a sequence of independent random variables with zero mean and common variance σ^2. If $X_n = Y_1 + \cdots + Y_n$, then show that $X_n^2 - n\sigma^2$ is a martingale.

5. Let $X_0, \cdots X_n, \cdots$ be a sequence of independent random variables taking values $0, 1, 2$ with probabilities respectively, p_0, p_1, p_2. The process stops when a subsequence s' appears for the first time. Find the expected stopping time $E[T]$ in each of the following cases:

   (a) $s' \equiv 1, 2, 3$.

   (b) $s' \equiv 2, 2, 2$.

   (c) $s' \equiv 1, 2, 1$.

   (d) $s' \equiv 2, 1, 1$.

6. How would you solve the previous problem using the Markov chain ideas discussed in class?